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Mixture pressure and stress in disperse two-phase flow

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The senior author (AP) wishes to dedicate this study to Professor Gad Hetsroni on occasion of his 65th birthday. In spite of a first acquaintance in a seemingly inauspicious place, he had the good fortune of enjoying Professor Hetsroni's warm friendship and wise guidance on numerous occasions. Among the many that stand out in his memory is a phone call at a critical moment and an evening in Cesarea. Grazie Gad, grazie di tutto, e tantissimi auguri.

Abstract

The definition and interpretation of average pressure in an incompressible disperse two-phase flow are ambiguous and have been the object of debate in the literature. For example, the physical meaning of definitions involving an internal 'pressure' inside rigid particles is unclear. The appearance of the particle internal stresses in averaged equations of the two-fluid types is similarly puzzling as, provided the particles are sufficiently rigid, the precise numerical value of such stresses would not be expected to affect the flow. This paper deals with these matters using a new approach. A proper definition of mixture pressure follows quite naturally by identifying the isotropic component of the mixture stress that — just like the usual pressure in incompressible single-phase flow — is covariant under the gauge transformation $p \rightarrow p + \psi$, where ψ can be thought of as the potential of body forces. This transformation includes as special cases the more usual gauge transformation $p \rightarrow p + \Pi(t)$, with $\Pi(t)$ an arbitrary function of time, and $p \rightarrow p - \rho \mathbf{g} \cdot \mathbf{x}$, by which gravitational effects are removed from the single-phase equations. The mixture pressure that is identified on the basis of this argument contains the pressure averaged over the surface of the particles, as in some earlier proposals, but also other terms. Explicit examples are given for the case of dilute potential and Stokes flows of spheres. It is also shown that it is possible to completely eliminate the disperse-phase stress field from the averaged equations provided the particle motion is only expressed in terms of the center-of-mass and angular velocity. Finally, the implications for the closure of the averaged equations that derive from the concept of

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covariance under the general gauge transformation are discussed.

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1. Introduction

Consider a disperse two-phase flow consisting of drops or bubbles suspended in a continuous phase. If one wanted to define an average pressure in the mixture, the obvious choice would be the sum of the pressures in the two phases, each weighted according to its local volume fraction. Indeed, most averaging methods would lead to such a result. Let now the disperse phase become gradually more viscous, e.g., by decreasing the temperature. As long as it remained a fluid — however viscous — its internal pressure would be well defined and this definition of average pressure would be meaningful. However, when the viscosity is large enough, the behavior of the drops would be indistinguishable from that of rigid particles and yet, although the average flow would be exactly the same in the two cases, the concept of 'pressure' inside a rigid particle would be devoid of physical meaning.

Consider now the averaged momentum equation for the disperse phase. Again, as long as this phase consists of a fluid, most averaging methods would lead to a term involving the pressure gradient of the disperse phase. If the disperse phase were to become more and more viscous, however, we would encounter the same conceptual difficulty as before.

These are just two manifestations of the paradox associated with the understanding of disperse-phase pressure and mixture pressure in disperse two-phase flow, a situation that has spawned a considerable literature (see, e.g., Bouré, 1979; Prosperetti and Jones, 1984; Givler, 1993; Hwang and Shen, 1989; Joseph and Lundgren, 1990; Drew and Lahey, 1993).

Several authors avoid the introduction of a disperse-phase pressure and replace it by an 'interfacial pressure', related to the mean continuous-phase pressure in the neighborhood of the particles (see, e.g., Anderson and Jackson, 1967; Ishii, 1975; Drew, 1983; Prosperetti and Jones, 1984; Arnold et al., 1989). It will be shown that this concept is a good approximation to the complete solution to the problem that emerges from our study.

The problem associated with the concept of mixture pressure is of course intimately related to that of mixture stress. Here, again, the formal application of straightforward averaging causes the stress inside the particles to appear. To deal with the case of rigid particles, in principle one could start with the equations of elasticity inside the particles and consider the limit as the elastic modulus becomes larger and larger (Drew and Lahey, 1993). This is certainly an interesting approach but, provided the particles are sufficiently rigid, the precise numerical value of the stresses would not be expected to significantly affect the flow and one would think that a simpler approach would be adequate (Prosperetti and Zhang, 1996).

The purpose of the present paper is to present a unified discussion of these matters and offer a hopefully satisfactory resolution of the difficulties associated with them. At the same time, we discuss several aspects of the averaged momentum equations, particularly in the all-important case of spatially non-uniform systems, and show how a conceptually complete theory can be formulated without any reference to the particles' internal structure provided they are sufficiently stiff that they can be modeled as rigid. In this description the particle motion is characterized only in terms of the center-of-mass velocity and angular velocity.

Our starting point is the simple observation that a conservative body force characterized by a potential ψ can be eliminated from the exact microscopic equations for an incompressible fluid by subjecting the pressure to the gauge transformation (see, e.g., Batchelor, 1967)

$$p \to p + \psi. \tag{1}$$

We propose to *define* the mixture pressure as that part of the total stress in the mixture that transforms as in Eq. (1). It will be shown that this prescription leads to a unique result that is related to (but not identical with) earlier proposals in the literature and has the correct form in the case of dilute suspensions both at small and large particle Reynolds numbers. Sometimes, in Continuum Mechanics, pressure is related to the trace of the stress. We discuss this matter in Section 7 and point out the difficulties associated with it.

The problem that we discuss is also of great importance for the closure of averaged equations models and, indeed, we have been led to its consideration by our study of the closure problem (Marchioro et al., 1999a, 1999b). The point is that, in a two-fluid averaged equations model, the mixture pressure is part of the primary unknowns and can in principle be found by solving the equations. Closure relations must be provided, on the other hand, for the remaining part of the stress. It is therefore evident that the lack of a correct definition for the mixture pressure introduces serious uncertainties for the development of closure relations.

While a detailed understanding of the present paper requires some material from our earlier studies (Zhang and Prosperetti, 1994a, 1997; Prosperetti, 1998), the line of the argument can be followed also without such specialized knowledge. Therefore, in order to get as directly as possible to the results, we summarize the relevant background material in Appendix A giving references to the appropriate equations along the way. The averaged continuity equations have the standard form and are given explicitly in Eqs. (A11) and (A12) of Appendix A. We focus on the momentum equations.

The considerations and explicit results that follow refer to a suspension of equal spherical particles in an incompressible fluid. Extensions to unequal, or non-spherical, particles are conceptually straightforward, but would result in more cumbersome expressions.

2. Momentum balance

We write the microscopic momentum equation for the continuous phase (index C) in the form

$$\rho_C \left[\frac{\partial \mathbf{u}_C}{\partial t} + \nabla \cdot (\mathbf{u}_C \mathbf{u}_C) \right] = \nabla \cdot \boldsymbol{\sigma}_C - \nabla \psi_C, \tag{2}$$

where $\mathbf{u}_C$ is the velocity, ρ_C the density, $\boldsymbol{\sigma}_C$ the stress, and ψ_C the potential of the body force. In the case of gravity with an acceleration $\mathbf{g}$, for example, we would have $\psi_C = -\rho_C \mathbf{g} \cdot \mathbf{x}$. We shall only consider the case in which ψ_C is a prescribed field of force, not influenced by the flow itself.[2] Thus, the force field has no sources in the volume occupied by the fluid with the consequence that ψ_C is harmonic. Note explicitly that we do not make any assumption as to the constitutive relation of $\boldsymbol{\sigma}_C$. The results that follow are applicable to both Newtonian and non-Newtonian fluids.

Upon taking the phase ensemble average of (2) according to the rules summarized in the Appendix we find

$$\mathbf{I}_C = \beta_C \langle \nabla \cdot \boldsymbol{\sigma}_C \rangle - \beta_C \nabla \psi_C, \tag{3}$$

where angle brackets denote the phase-ensemble average, β_C is the volume fraction of the continuous phase, and we denote the inertia terms in the left-hand side by $\mathbf{I}_C$ for brevity:

$$\mathbf{I}_C = \rho_C \frac{\partial}{\partial t}\big(\beta_C \langle \mathbf{u}_C \rangle\big) + \rho_C \nabla \cdot \big(\beta_C \langle \mathbf{u}_C \mathbf{u}_C \rangle\big); \tag{4}$$

here and throughout the paper we assume both phases to be incompressible.

At this point one faces the well-known problem that differentiation and averaging do not commute. Indeed, according to the relations developed in Zhang and Prosperetti (1994a, 1997), (see also Prosperetti, 1998), one has

$$\beta_C \langle \nabla \cdot \boldsymbol{\sigma}_C \rangle = \nabla \cdot \big(\beta_C \langle \boldsymbol{\sigma}_C \rangle\big) - \int_{|\mathbf{x}-\mathbf{y}|=a} \mathrm{d}S_y P(\mathbf{y}) \langle \boldsymbol{\sigma}_C \rangle_1 (\mathbf{x}|\mathbf{y}) \cdot \mathbf{n}_y, \tag{5}$$

where $P(\mathbf{y})$ is the single-particle probability density defined in Eq. (A7) and $\langle \boldsymbol{\sigma}_C \rangle_1 (\mathbf{x}|\mathbf{y})$ is the stress at $\mathbf{x}$ averaged conditionally (see the definition (A8)) to the presence of a particle with center at $\mathbf{y}$. For brevity, here and in the following, we omit the explicit indication of all non-essential variables such as time, and leave some of the integrations over the probability variables implicit. For example, if integration over the velocity variable were explicitly indicated, Eq. (5) would take on the form given in Eq. (A16) of Appendix A.

The integral in Eq. (5) is effected over the centers of all the spherical particles at a distance equal to the radius a from the field point $\mathbf{x}$ under consideration. As explained in our earlier papers, we can put Eq. (5) in a more convenient form by carrying out a procedure we refer to as *small-particle approximation*. The idea is that the spatial scales over which $\langle \boldsymbol{\sigma}_C \rangle_1 (\mathbf{x}|\mathbf{y})$ varies with respect to the arguments $\mathbf{x}$ and $\mathbf{y}$ are very different. Near the particle, the scale of variation with respect to the variable $\mathbf{x}$ is comparable to — or even smaller than — the particle radius a. On the other hand, the scale of variation with respect to the position $\mathbf{y}$ of the particle center, with $|\mathbf{x}-\mathbf{y}| = a$ fixed, is of the order of the macroscopic length scale L which, in the bulk of the suspension, is normally much greater than a. The scale of variation of the single-particle probability distribution $P(\mathbf{y})$ is also the macroscopic length scale. By exploiting this idea in the manner described in Appendix B, we find

$$\beta_C \langle \nabla \cdot \boldsymbol{\sigma}_C \rangle = \nabla \cdot \big(\beta_C \langle \boldsymbol{\sigma}_C \rangle\big) - n\mathscr{A}[\boldsymbol{\sigma}_C] + \nabla \cdot \big(\beta_D \mathscr{L}[\boldsymbol{\sigma}_C]\big), \tag{6}$$

[2] A counter-example would be the self-gravitational field in the interior of a celestial body or the electric field in a plasma.

where n is the particle number density defined in Eq. (A5) and

$$\beta_{\rm D}\mathscr{L}[\boldsymbol{\sigma}_{\rm C}] = n\mathscr{T}[\boldsymbol{\sigma}_{\rm C}] + \nabla\cdot\left\{n\mathscr{S}[\boldsymbol{\sigma}_{\rm C}] + \nabla\cdot\left[n\mathscr{R}[\boldsymbol{\sigma}_{\rm C}] + \cdots\right]\right\}, \tag{7}$$

with

$$\mathscr{A}[\boldsymbol{\sigma}_{\rm C}](\mathbf{x}) = \overline{\int_{|\mathbf{r}|=a} {\rm d}S_r \boldsymbol{\sigma}_{\rm C}(\mathbf{x}+\mathbf{r}|\mathbf{x},N-1)\cdot\mathbf{n}}, \tag{8}$$

$$\mathscr{T}[\boldsymbol{\sigma}_{\rm C}](\mathbf{x}) = a\overline{\int_{|\mathbf{r}|=a} {\rm d}S_r \mathbf{n}\left[\boldsymbol{\sigma}_{\rm C}(\mathbf{x}+\mathbf{r}|\mathbf{x},N-1)\cdot\mathbf{n}\right]}, \tag{9}$$

$$\mathscr{S}[\boldsymbol{\sigma}_{\rm C}](\mathbf{x}) = -\frac{1}{2}a^2\overline{\int_{|\mathbf{r}|=a} {\rm d}S_r \mathbf{n}\mathbf{n}\left[\boldsymbol{\sigma}_{\rm C}(\mathbf{x}+\mathbf{r}|\mathbf{x},N-1)\cdot\mathbf{n}\right]}, \tag{10}$$

$$\mathscr{R}[\boldsymbol{\sigma}_{\rm C}](\mathbf{x}) = \frac{1}{6}a^3\overline{\int_{|\mathbf{r}|=a} {\rm d}S_r \mathbf{n}\mathbf{n}\mathbf{n}\left[\boldsymbol{\sigma}_{\rm C}(\mathbf{x}+\mathbf{r}|\mathbf{x},N-1)\cdot\mathbf{n}\right]}. \tag{11}$$

Here the overline denotes the particle average defined in Eq. (A9), i.e., the ensemble average over all the configurations such that one of the particles has center at $\mathbf{x}$; the integration is over the surface of that particle. The terms neglected in Eq. (7) are of higher order in a/L. One recognizes that, in particular, $\mathscr{A}$ is the average hydrodynamic force on the particles with centers contained in the unit volume. Approximately, the particle number density is related to the disperse-phase volume fraction $\beta_{\rm D}$ by

$$\beta_{\rm D} = \left(1 + \frac{a^2}{10}\nabla^2 + \cdots\right)(nv), \tag{12}$$

where $v = \frac{4}{3}\pi a^3$ is the volume of each particle and the omitted terms are $o(a/L)^2$. Note that $\beta_{\rm D} = nv$ only when the particle distribution is spatially uniform. With Eq. (6) the averaged momentum equation (3) becomes

$$\mathbf{I}_{\rm C} = \nabla\cdot\left(\beta_{\rm C}\langle\boldsymbol{\sigma}_{\rm C}\rangle + \beta_{\rm D}\mathscr{L}[\boldsymbol{\sigma}_{\rm C}]\right) - n\mathscr{A}[\boldsymbol{\sigma}_{\rm C}] - \beta_{\rm C}\nabla\psi_{\rm C}. \tag{13}$$

For the disperse phase we write the microscopic equation of motion as

$$\rho_{\rm D}\mathbf{a}_{\rm D} = \nabla\cdot\boldsymbol{\sigma}_{\rm D} - \nabla\psi_{\rm D}, \tag{14}$$

where $\mathbf{a}_{\rm D}$ and $\rho_{\rm D}$ are the acceleration and density of the particle material, $\boldsymbol{\sigma}_{\rm D}$ is the stress tensor, and $\psi_{\rm D}$ is the potential of the body force, also taken to be harmonic. We take the phase average and find

$$\mathbf{I}_{\rm D} = \beta_{\rm D}\langle\nabla\cdot\boldsymbol{\sigma}_{\rm D}\rangle - \beta_{\rm D}\nabla\psi_{\rm D}, \tag{15}$$

where $\mathbf{I}_{\rm D}$ denotes the inertia terms and is defined as in Eq. (4). As shown in Appendix B, for the disperse phase one can develop a small-particle approximation analogous to Eq. (6), which puts the equation in the form

$$\mathbf{I}_\mathrm{D} = n\mathscr{A}_\mathrm{D}[\boldsymbol{\sigma}_\mathrm{C}] + \nabla \cdot \boldsymbol{\Sigma}_\mathrm{a} - \beta_\mathrm{D}\nabla\psi_\mathrm{D}, \tag{16}$$

where $\mathscr{A}_\mathrm{D}$ is defined by

$$\mathscr{A}_\mathrm{D}[\boldsymbol{\sigma}_\mathrm{D}](\mathbf{x}) = \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r \boldsymbol{\sigma}_\mathrm{D}(\mathbf{x}+\mathbf{r}|\mathbf{x}, N-1) \cdot \mathbf{n}}. \tag{17}$$

Since, at the particle surface,

$$\boldsymbol{\sigma}_\mathrm{C} \cdot \mathbf{n} = \boldsymbol{\sigma}_\mathrm{D} \cdot \mathbf{n}, \tag{18}$$

we recognize that $\mathscr{A}_\mathrm{D} = \mathscr{A}$. The stress $\boldsymbol{\Sigma}_\mathrm{a}$ is conceptually similar to $\mathscr{L}$; its explicit expression is

$$\boldsymbol{\Sigma}_\mathrm{a} = -n(\mathbf{x}) \int_{|\mathbf{r}| \leq a} \mathrm{d}^3 r \ \mathbf{r}\langle \nabla_r \cdot \boldsymbol{\sigma}_\mathrm{D} \rangle_1 (\mathbf{x}+\mathbf{r}|\mathbf{x}) + \frac{1}{2}\nabla \cdot \left[n(\mathbf{x}) \int_{|\mathbf{r}| \leq a} \mathrm{d}^3 r \ \mathbf{r}\mathbf{r}\langle \nabla_r \boldsymbol{\sigma}_\mathrm{D} \rangle_1 (\mathbf{x}+\mathbf{r}|\mathbf{x}) \right] + \cdots \tag{19}$$

In this relation $\langle \nabla_r \cdot \boldsymbol{\sigma}_\mathrm{D} \rangle_1 (\mathbf{x} + \mathbf{r}|\mathbf{x})$ is the ensemble average of $\nabla_r \cdot \boldsymbol{\sigma}_\mathrm{D}$ calculated with one particle fixed at **x**; a precise definition is given in Eq. (A8).

Upon adding the two averaged equations (13) and (16), we find the total mixture momentum balance in the form

$$\mathbf{I}_\mathrm{C} + \mathbf{I}_\mathrm{D} = \nabla \cdot \left(\beta_\mathrm{C} \langle \boldsymbol{\sigma}_\mathrm{C} \rangle + \beta_\mathrm{D} \mathscr{L}[\boldsymbol{\sigma}_\mathrm{C}] + \boldsymbol{\Sigma}_\mathrm{a} \right) - \beta_\mathrm{C}\nabla\psi_\mathrm{C} - \beta_\mathrm{D}\nabla\psi_\mathrm{D}. \tag{20}$$

This expression conforms to the expectation that, aside from the body forces, the only source of momentum for the mixture can be written as the divergence of a tensor, which expresses the fact that a volume element interacts with its surroundings only through surface forces. The tensor under the divergence sign may be identified with the total mixture stress $\boldsymbol{\Sigma}_\mathrm{T}$:

$$\boldsymbol{\Sigma}_\mathrm{T} = \beta_\mathrm{C} \langle \boldsymbol{\sigma}_\mathrm{C} \rangle + \beta_\mathrm{D} \mathscr{L}[\boldsymbol{\sigma}_\mathrm{C}] + \boldsymbol{\Sigma}_\mathrm{a}. \tag{21}$$

In view of the following developments, it is important to note that, both from the manner of its derivation and from the way it appears in Eq. (20), the tensor $\boldsymbol{\Sigma}_\mathrm{T}$ is only defined up to a divergenceless term.

3. Comparison with other forms of the momentum equations

Before proceeding, it may be useful to establish explicitly the connection between the two averaged momentum equations (13) and (16) and the standard form of the momentum equations in the so-called two-fluid model. In addition to demonstrating the substantial identity of the two formulations, this analysis leads to some insight into the nature of the interphase forces.

In the two-fluid model the momentum equations are usually written as

$$\mathbf{I}_C = \beta_C \nabla \cdot \mathbf{S}_C - \mathbf{F} - \beta_C \nabla \psi_C, \tag{22}$$

$$\boldsymbol{I}_D = \beta_D \nabla \cdot \mathbf{S}_D + \mathbf{F} - \beta_D \nabla \psi_D, \tag{23}$$

where $\mathbf{S}_{C,D}$ are suitable stresses and the phase-interaction force $\mathbf{F}$ appears with opposite signs in the two equations so as to explicitly satisfy the action–reaction principle. Upon adding these two equations the result is compatible with Eq. (20) only if

$$\beta_C \nabla \cdot \mathbf{S}_C + \beta_D \nabla \cdot \mathbf{S}_D = \nabla \cdot \boldsymbol{\Sigma}_T, \tag{24}$$

which, using $\beta_C = 1 - \beta_D$, may be rewritten as

$$\beta_D \nabla \cdot (\mathbf{S}_D - \mathbf{S}_C) = \nabla \cdot (\boldsymbol{\Sigma}_T - \mathbf{S}_C). \tag{25}$$

This equation must express a formal identity and not a physical law, in the sense that it must be satisfied identically independently of any specific flow. This requires that the left-hand side be also expressible as a divergence, which is only possible if the difference $\mathbf{S}_D - \mathbf{S}_C$ is a function of β_D only. If this were so, in a system at rest and not subject to forces, one would have a difference between the phase stresses only due to the distribution of the particles, which is physically unreasonable. We thus conclude that

$$\mathbf{S}_C = \mathbf{S}_D = \boldsymbol{\Sigma}_T, \tag{26}$$

possibly up to a divergenceless tensor that, as already noted, has no physical consequences. Upon substituting into Eq. (22) or (23) and comparing with Eq. (13) or (16) we then have

$$\mathbf{F} = n\mathscr{A} - \beta_D \nabla \cdot \boldsymbol{\Sigma}_T + \nabla \cdot \boldsymbol{\Sigma}_a. \tag{27}$$

Upon extracting the term $\boldsymbol{\Sigma}_a$ from $\boldsymbol{\Sigma}_T$, it readily follows from this relation that $\mathbf{F} - \beta_C \nabla \cdot \boldsymbol{\Sigma}_a$ is independent of the particle stress.

With the previous results (26) and (27), the two momentum equations (22) and (23) take on the symmetric forms

$$\mathbf{I}_C = \beta_C \nabla \cdot \boldsymbol{\Sigma}_T - \mathbf{F} - \beta_C \nabla \psi_C, \tag{28}$$

$$\mathbf{I}_D = \beta_D \nabla \cdot \boldsymbol{\Sigma}_T + \mathbf{F} - \beta_D \nabla \psi_D. \tag{29}$$

It is interesting to note that, given that $\beta_C \nabla \cdot \boldsymbol{\Sigma}_a - \mathbf{F}$ is independent of $\boldsymbol{\Sigma}_a$, the continuous-phase momentum equation does not explicitly depend on the inner particle dynamics as expected.

4. Gauge transformation

It is a fundamental property of the momentum equation of an incompressible fluid that an external body force can be removed by a gauge transformation of the isotropic part of the stress. If we let

$$\hat{\boldsymbol{\sigma}}_C = \boldsymbol{\sigma}_C - \psi \mathbf{I}, \tag{30}$$

where $\mathbf{I}$ is the identity two-tensor and ψ a harmonic function, the momentum equation (2) becomes:

$$\rho_C\left[\frac{\partial \mathbf{u}_C}{\partial t}+\nabla\cdot(\mathbf{u}_C\mathbf{u}_C)\right]=\nabla\cdot\hat{\boldsymbol{\sigma}}_C-\nabla\left(\psi_C-\psi\right). \tag{31}$$

In particular, if $\psi=\psi_C$, the effect of the body force is removed from the continuous phase. This simple property embodies the fundamental physical fact that an incompressible fluid responds to pressure gradients, without distinguishing between those due to a body force or other agents. Upon defining

$$\hat{\boldsymbol{\sigma}}_D=\boldsymbol{\sigma}_D-\psi\mathbf{I}, \tag{32}$$

the microscopic disperse phase momentum equation can also be brought to a similar form. Note that the same condition (18) on the normal stresses at the particle surfaces is satisfied by $\hat{\boldsymbol{\sigma}}_C$ and $\hat{\boldsymbol{\sigma}}_D$.

We wish to break up the total mixture stress (21) into a component to be identified with the mixture pressure p_m and a component, $\boldsymbol{\Sigma}$, due to viscous forces:

$$\boldsymbol{\Sigma}_T=-p_m\mathbf{I}+\boldsymbol{\Sigma}. \tag{33}$$

As a guide to the correct way by which to effect this decomposition, it seems natural to require that the fundamental gauge transformation property of the single-phase pressure be enjoyed also by p_m. This condition is consistent with the point of view of the two-fluid model according to which the two phases are to be considered as two interpenetrating fluid continua. We thus seek the part of $\boldsymbol{\Sigma}_T$ that transforms according to

$$\hat{\boldsymbol{\Sigma}}_T=\Sigma_T-\psi\mathbf{I}, \tag{34}$$

so that, upon carrying out the gauge transformation, the total momentum equation (20) becomes

$$\mathbf{I}_C+\mathbf{I}_D=\nabla\cdot\left(-\hat{p}_m+\boldsymbol{\Sigma}\right)-\beta_C\nabla\left(\psi_C-\psi\right)-\beta_D\nabla\left(\psi_D-\psi\right). \tag{35}$$

Note that here we write $\boldsymbol{\Sigma}$ rather than $\hat{\boldsymbol{\Sigma}}$ as one would expect the viscous part of the stress to be unaffected by the transformation, precisely as in the case of a single-phase incompressible fluid.

It is easy to show from the definition (19) of $\boldsymbol{\Sigma}_a$, the transformation relation (42), and the fact that ψ is harmonic, that

$$\hat{\boldsymbol{\Sigma}}_a=\boldsymbol{\Sigma}_a+\frac{va^2}{10}\left[n\partial_i\partial_j\psi-\left(\partial_j n\right)\left(\partial_i\psi\right)\right]-\frac{va^4}{70}\left(\partial_k n\right)\left(\partial_k\partial_j\partial_i\psi\right)+\cdots \tag{36}$$

from which

$$\nabla\cdot\left(\boldsymbol{\Sigma}_a-\hat{\boldsymbol{\Sigma}}_a\right)=\left(\beta_D-nv\right)\nabla\psi+\text{higher order terms}. \tag{37}$$

(Both of these relations exhibit error terms because $\boldsymbol{\Sigma}_a$ is defined by a perturbation expansion.) Furthermore, from its definition (8), we find that $\mathscr{A}$ transforms according to

$$\hat{\mathscr{A}} = \mathscr{A} - v\nabla\psi. \tag{38}$$

With these relations the continuous-phase average momentum equation (13) is readily found to become

$$\mathbf{I}_\mathrm{C} = \nabla\cdot\left(\hat{\boldsymbol{\Sigma}}_\mathrm{T} - \hat{\boldsymbol{\Sigma}}_\mathrm{a}\right) - n\hat{\mathscr{A}} - \beta_\mathrm{C}\nabla\left(\psi_\mathrm{C} - \psi\right), \tag{39}$$

while the disperse-phase equation (16) transforms to

$$\mathbf{I}_\mathrm{D} = n\hat{\mathscr{A}} + \nabla\cdot\hat{\boldsymbol{\Sigma}}_\mathrm{a} - \beta_\mathrm{D}\nabla\left(\psi_\mathrm{D} - \psi\right). \tag{40}$$

As an example one may consider the case of gravity for which $\psi_{\mathrm{C,D}} = -\rho_{\mathrm{C,D}}\mathbf{g}\cdot\mathbf{x}$. If one were to take $\psi = \psi_\mathrm{C}$, one would remove the body force from both the microscopic and the average continuous phase momentum equations, while the disperse-phase equation would acquire a term $\beta_\mathrm{D}(\rho_\mathrm{D} - \rho_\mathrm{C})\mathbf{g}$, i.e., the Archimedean force.

In the following it is not necessary to commit oneself to a specific choice for ψ although, in many practical cases, $\psi = \psi_\mathrm{C}$ would be the physically significant gauge transformation.

Before concluding this section it is useful to derive a few other relations concerning the gauge transformation properties of some other quantities involved in the theory.

Averaging and the small-particle approximation (6) applied to (31) give

$$\mathbf{I}_\mathrm{C} = \nabla\cdot\left(\beta_\mathrm{C}\langle\hat{\boldsymbol{\sigma}}_\mathrm{C}\rangle + \beta_\mathrm{D}\hat{\mathscr{L}}\right) - n\hat{\mathscr{A}} - \beta_\mathrm{C}\nabla\left(\psi_\mathrm{C} - \psi\right). \tag{41}$$

Upon subtracting from the untransformed momentum equation (13) and using Eqs. (30) and (38) one readily finds that

$$\nabla\cdot\left(\beta_\mathrm{D}\mathscr{L}\right) = \nabla\cdot\left(\beta_\mathrm{D}\hat{\mathscr{L}}\right) + \psi\nabla\beta_\mathrm{D} + nv\nabla\psi = \nabla\cdot\left(\beta_\mathrm{D}\hat{\mathscr{L}} + \beta_\mathrm{D}\psi\right) - \left(\beta_\mathrm{D} - nv\right)\nabla\psi, \tag{42}$$

which can be proved directly from the definition of the various quantities involved provided terms of higher order than those retained in $\mathscr{L}$ are consistently discarded. Upon applying the gauge transformation to the interphase force $\mathbf{F}$ defined in Eq. (27) we then find

$$\mathbf{F} = n\hat{\mathscr{A}} - \beta_\mathrm{D}\nabla\cdot\left(\beta_\mathrm{C}\langle\hat{\boldsymbol{\sigma}}_\mathrm{C}\rangle + \beta_\mathrm{D}\hat{\mathscr{L}} + \hat{\boldsymbol{\Sigma}}_\mathrm{a}\right) + \beta_\mathrm{C}\left[\nabla\cdot\left(\boldsymbol{\Sigma}_\mathrm{a} - \hat{\boldsymbol{\Sigma}}_\mathrm{a}\right) + \left(nv - \beta_\mathrm{D}\right)\nabla\psi\right], \tag{43}$$

or, from Eq. (37), $\hat{\mathbf{F}} = \mathbf{F}$. We thus conclude that $\mathbf{F}$ is gauge invariant, as one would expect from a quantity to be identified with the interphase force.[3]

5. The mixture pressure

Eq. (36) shows that the transformation property of $\boldsymbol{\Sigma}_\mathrm{a}$ does not include a scalar. Therefore,

[3] Sometimes the added mass force is expressed in terms of a pressure gradient, but this form is only found upon using the liquid momentum equation to eliminate the acceleration. If the force is expressed in terms of the liquid acceleration, gauge invariance is manifest.

in order to achieve the transformation property (34) of $\boldsymbol{\Sigma}_T$, it is sufficient to focus on the continuous phase stress $\beta_C\langle\boldsymbol{\sigma}_C\rangle + \beta_D\mathscr{L}$. The transformation property of the first term is simply

$$\beta_C\langle\hat{\boldsymbol{\sigma}}_C\rangle = \beta_C\langle\boldsymbol{\sigma}_C\rangle - \beta_C\psi\mathbf{I}. \tag{44}$$

The crucial term is the second one, $\beta_D\mathscr{L}$. Since $\beta_D\mathscr{L}$ is given in Eq. (7) as a perturbation series, we examine the individual terms one by one. At each step we separate the part of the term considered which, upon effecting the gauge transformation, is not capable of giving an isotropic contribution to $\boldsymbol{\Sigma}_T$. It will be easy to find the correct form of p_m by inspecting the remaining parts.

5.1. The transformation of $\mathscr{T}$

Using the result, valid for a harmonic function,

$$a\int_{|\mathbf{r}|=a} \mathrm{d}S_r\mathbf{nn}\psi = v\left(I + \frac{a^2}{5}\nabla\nabla\right)\psi, \tag{45}$$

it is a simple matter to find

$$\mathscr{T} = \hat{\mathscr{T}} + v\mathbf{I}\psi + \frac{a^2}{5}v\nabla\nabla\psi \tag{46}$$

Since $\nabla^2\psi = 0$, we also have

$$\mathrm{Tr}\mathscr{T} = \mathrm{Tr}\,\hat{\mathscr{T}} + 3v\psi. \tag{47}$$

Therefore, we write

$$\mathscr{T} = \mathscr{T}^0 + \frac{1}{3}\mathbf{I}\,\mathrm{Tr}\,\mathscr{T}, \tag{48}$$

where the last term is clearly isotropic. The new quantity $\mathscr{T}^0$ transforms according to

$$\mathscr{T}^0 = \hat{\mathscr{T}}^0 + \frac{a^2}{5}v\nabla\nabla\psi, \tag{49}$$

and, since it does not contain an isotropic part, it cannot contribute to p_m. From the definition (9) of $\mathscr{T}$, in terms of particle averages we have

$$\mathscr{T}^0 = \overline{a\int_{|\mathbf{r}|=a} \mathrm{d}S_r\left[\mathbf{n}(\boldsymbol{\sigma}_C\cdot\mathbf{n}) - \frac{1}{3}\mathbf{I}(\mathbf{n}\cdot\boldsymbol{\sigma}_C\cdot\mathbf{n})\right]}. \tag{50}$$

It is readily checked that the symmetric part of $\mathscr{T}^0$ is just the stresslet, or force dipole:

$$t^s_{ij} = \frac{1}{2}\left(\mathscr{T}^0_{ij} + \mathscr{T}^0_{ji}\right) = \overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r\left[\frac{1}{2}\left(r_i(\boldsymbol{\sigma}_C)_{jk} + r_j(\boldsymbol{\sigma}_C)_{ik}\right) - \frac{1}{3}\delta_{ij}(\mathbf{r}\cdot\boldsymbol{\sigma}_C)_k\right]n_k}, \tag{51}$$

while the antisymmetric part is the rotlet:

$$t^{\mathrm{a}}_{ij} = \frac{1}{2}\left(\mathscr{T}^0_{ij} - \mathscr{T}^0_{ji}\right) = \epsilon_{ijk}\left(\overline{\frac{1}{2}\int_{|\mathbf{r}|=a} \mathrm{d}S_r(\boldsymbol{\sigma}_{\mathrm{C}}\cdot\mathbf{n})\times\mathbf{r}}\right)_k. \tag{52}$$

5.2. The transformation of $\mathscr{S}$

With the result

$$\int_{|\mathbf{r}|=a} \mathrm{d}S_r n_i n_j n_k \psi = \frac{v}{5}\left(\delta_{ijkl}\partial_l\psi + \frac{a^2}{7}\partial_i\partial_j\partial_k\psi\right), \tag{53}$$

where

$$\delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} \tag{54}$$

is the fourth-order completely symmetric tensor, we readily find

$$\mathscr{S} = \hat{\mathscr{S}} - \frac{va^2}{10}\left(\delta_{ijkl}\partial_l\psi + \frac{a^2}{7}\partial_i\partial_j\partial_k\psi\right). \tag{55}$$

Since $\mathscr{S}$ enters the stress as a divergence, the first term in parentheses can give an isotropic contribution and must be removed. For this purpose we note that, from Eq. (38), $\mathscr{A}$ exhibits the correct transformation properties and we therefore set

$$\mathscr{S}^0 = \mathscr{S} + \frac{a^2}{10}\delta_{ijkl}\mathscr{A}_l, \tag{56}$$

from which

$$\mathscr{S}^0 = \hat{\mathscr{S}}^0 - \frac{a^4}{70}\partial_i\partial_j\partial_k\psi. \tag{57}$$

In terms of particle averages

$$\mathscr{S}^0_{kji} = -\frac{1}{2}a^2\overline{\int_{|\mathbf{r}|=a}\mathrm{d}S_r\left[n_k n_j(\boldsymbol{\sigma}_{\mathrm{C}}\cdot\mathbf{n})_i - \frac{1}{5}\left(\delta_{ij}(\mathbf{n}\cdot\boldsymbol{\sigma}_{\mathrm{C}})_k + \delta_{kj}(\mathbf{n}\cdot\boldsymbol{\sigma}_{\mathrm{C}})_i + \delta_{ki}(\mathbf{n}\cdot\boldsymbol{\sigma}_{\mathrm{C}})_j\right)\right]}. \tag{58}$$

As before, we can decompose $\mathscr{S}^0_{kji}$ into a traceless symmetric part, an antisymmetric part, and an isotropic part:

$$\mathscr{s}^{\mathrm{s}}_{kji} = \frac{1}{2}\left(\mathscr{S}^0_{kji} + \mathscr{S}^0_{kij}\right) - \frac{1}{3}\delta_{ij}\mathscr{S}^0_{kmm}, \tag{59}$$

$$\mathscr{s}^{\mathrm{a}}_{kji} = \frac{1}{2}\left(\mathscr{S}^0_{kji} - \mathscr{S}^0_{kij}\right) = \epsilon_{ijm}\left(\overline{\frac{1}{4}\int_{|\mathbf{r}|=a}\mathrm{d}S_r r_k\left[(\boldsymbol{\sigma}_{\mathrm{C}}\cdot\mathbf{n})\times\mathbf{r}\right]_m}\right), \tag{60}$$

$$s^{\mathrm{i}}_{kmm} = \frac{1}{3}\mathscr{S}^0_{kmm} = -\frac{1}{2}\overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r\Big[r_k(\mathbf{r}\cdot\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n}) - a^2(\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n})_k\Big]}. \tag{61}$$

5.3. The transformation of $\mathscr{R}$

To proceed similarly with $\mathscr{R}$ we need the result, valid for a harmonic ψ,

$$a\int_{|\mathbf{r}|=a} \mathrm{d}S_r n_i n_j n_k n_l \psi = \frac{1}{5}v\left(\delta_{ijkl}\psi + \frac{a^2}{14}\delta_{ijklpq}\partial_p\partial_q\psi + \frac{a^4}{63}\partial_i\partial_j\partial_k\partial_l\psi\right), \tag{62}$$

where δ_{ijkl} is defined in Eq. (54) and δ_{ijklpq} is the analogous sixth-order tensor. We therefore have

$$\mathscr{R}_{lkji} = \hat{\mathscr{R}}_{lkji} + \frac{a^2 v}{30}\left(\delta_{ijkl}\psi + \frac{a^2}{14}\delta_{ijklpq}\partial_p\partial_q\psi + \frac{a^4}{63}\partial_i\partial_j\partial_k\partial_l\psi\right). \tag{63}$$

From Eq. (47) we see that the first term in parentheses transforms like Tr $\mathscr{T}$ while, from Eq. (49), the second term transforms like $\mathscr{T}^0$. We therefore define

$$\mathscr{R}_{lkji} = \mathscr{R}^0_{lkji} + \frac{a^2}{84}\delta_{ijklpq}\mathscr{T}^0_{pq} + \frac{a^2}{90}\delta_{ijkl}Tr\,\mathscr{T}, \tag{64}$$

and note that

$$\mathscr{R}^0_{lkji} = \hat{\mathscr{R}}^0_{lkji} + \frac{a^6 v}{1890}\partial_i\partial_j\partial_k\partial_l\psi. \tag{65}$$

An expression for $\mathscr{R}^0$ in terms of particle averages is readily written down from Eqs. (9) and (11) but, since it is somewhat complicated, we do not show it.

The decomposition into a traceless symmetric part, an antisymmetric part, and an isotropic part is now

$$r^{\mathrm{s}}_{lkji} = \frac{1}{2}\left(\mathscr{R}^0_{lkji} + \mathscr{R}^0_{lkij}\right) - \frac{1}{3}\delta_{ij}\mathscr{R}^0_{lkmm}, \tag{66}$$

$$r^{\mathrm{a}}_{lkji} = \frac{1}{2}\left(\mathscr{R}^0_{lkji} - \mathscr{R}^0_{lkij}\right) = \epsilon_{ijm}\left(\frac{1}{12}\overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r r_l r_k\big[(\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n})\times\mathbf{r}\big]_m}\right). \tag{67}$$

$$r^{\mathrm{i}}_{lkmm} = \frac{1}{3}\mathscr{R}^0_{lkmm} = \frac{1}{18}\left[\overline{\int_{|\mathbf{r}|=a} \mathrm{d}S_r r_l r_k(\mathbf{r}\cdot\boldsymbol{\sigma}_\mathrm{C}\cdot\mathbf{n})} - a^2 t^{\mathrm{s}}_{lk} - \frac{a^2}{3}\delta_{lk}\,\mathrm{Tr}\,\mathscr{T}\right] \tag{68}$$

5.4. Final form of the stress

From Eqs. (6) and (7) we see that, ultimately, it is $\partial_j\partial_k\mathscr{S}_{kji}$ and $\partial_j\partial_l\partial_k\mathscr{R}_{lkji}$ that enter in the momentum equation, rather than $\mathscr{S}$, $\mathscr{R}$ themselves. We are thus at liberty to add a divergenceless tensor to $\beta_D\mathscr{L}$ without affecting the physics. With an eye to the symmetry properties of the final result it proves convenient to add

$$\frac{a^2}{10}(\delta_{jki}-\delta_{ijk})(n\mathscr{A}_k)+\frac{a^2}{45}(\partial_i\partial_j-\delta_{ij}\nabla^2)(n\,\mathrm{Tr}\,\mathscr{T})+\frac{a^2}{42}\Big\{(\partial_j\partial_k-\delta_{jk}\nabla^2)\big[n\big(\mathscr{T}^0_{ik}+\mathscr{T}^0_{ki}\big)\big]$$
$$+(\delta_{jl}\partial_i-\delta_{ij}\partial_l)\partial_k\big[n\big(\mathscr{T}^0_{kl}+\mathscr{T}^0_{lk}\big)\big]\Big\}, \tag{69}$$

with which we find

$$\beta_D\mathscr{L}=\delta_{ij}\left[\frac{1}{3}\left(1+\frac{a^2}{10}\nabla^2\right)(n\,\mathrm{Tr}\,\mathscr{T})-\frac{a^2}{5}\partial_k(n\mathscr{A}_k)+\frac{a^2}{14}\partial_k\partial_l\big(n\mathscr{T}^0_{kl}\big)\right]$$
$$+n\left[\mathscr{T}^0_{ij}-\frac{a^2}{10}(\partial_j\mathscr{A}_i+\partial_i\mathscr{A}_j)\right]+\frac{a^2}{28}\nabla^2\left[n\left(\mathscr{T}^0_{ij}+\mathscr{T}^0_{ji}\right)\right]$$
$$+\partial_k\left[n\mathscr{S}^0_{kji}+\partial_l\left(n\mathscr{R}^0_{lkji}\right)+\cdots\right]+\frac{a^2}{10}\big[n\partial_i\mathscr{A}_j-\mathscr{A}_i(\partial_j n)\big], \tag{70}$$

where the terms that have been dropped are of formal order $(a/L)^3$ and, therefore, of the same order as those dropped in the definition (7) of $\mathscr{L}$.

By means of Eqs. (47) and (38), and (48) it is readily verified that the quantity in brackets in the first line transforms according to

$$[\cdots]\to[\cdots]+\beta_D\psi-\frac{\nu a^4}{70}\partial_k\partial_l(n\partial_k\partial_l\psi). \tag{71}$$

The last term is of formal order $(a/L)^4$ and, therefore, is consistently negligible to the present order of approximation. With the transformation property (44) of $\langle\boldsymbol{\sigma}_C\rangle$, we may thus conclude that

$$\beta_C\langle\boldsymbol{\sigma}_C\rangle+\left[\frac{1}{3}\left(1+\frac{a^2}{10}\nabla^2\right)(n\,\mathrm{Tr}\,\mathscr{T})-\frac{a^2}{5}\partial_k(n\mathscr{A}_k)+\frac{a^2}{14}\partial_k\partial_l\big(n\mathscr{T}^0_{kl}\big)\right]\mathbf{I}, \tag{72}$$

transforms precisely as required by Eq. (34). This cannot yet be identified with the mixture pressure as it contains viscous contributions that should be removed. From the definitions (8) and (9), and writing the isotropic part of $\boldsymbol{\sigma}_C$ as $-p_C$, the pressure part is found to be

$$p_m=\beta_C\langle p_C\rangle+\left(1+\frac{a^2}{10}\nabla^2\right)\left(n\nu\bar{p}^e\right)+\frac{a^2}{5}\nabla\cdot\left(n\overline{\int_{|\mathbf{r}|=a}\mathrm{d}S_r(-p_C)\mathbf{n}}\right)$$
$$+\frac{a^2}{14}\nabla\nabla:\left[n\overline{\int_{|\mathbf{r}|=a}\mathrm{d}S_r\left(\mathbf{nn}-\frac{1}{3}\mathbf{I}\right)p_C}\right], \tag{73}$$

where p^e is the surface-average of the continuous-phase pressure over the particle surface:

$$p^e = \frac{1}{4\pi a^2}\int_{|\mathbf{r}|=a} \mathrm{d}S_r p_C. \tag{74}$$

Eq. (73) is the major result of this paper. We present a discussion in Section 6 and some examples in Section 8.

Returning to Eq. (70) we now identify, from the remaining terms, the invariant part that will contribute to the mixture viscous stress $\boldsymbol{\Sigma}$. It is easy to show that the four terms following the isotropic one are individually invariant up to $O(a/L)^3$ included. Of all the terms in Eq. (70), therefore, only the two in the last line are not invariant. Combining with the $n\mathscr{A}$ in Eq. (13) we find

$$-n\mathscr{A}_i + \partial_j\left[\frac{a^2}{10}\left(n\partial_i\mathscr{A}_j - (\partial_j n)\mathscr{A}_i\right)\right] = -\frac{\beta_D}{\nu}\mathscr{A}_i + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times\mathscr{A}) + n\nabla(\nabla\cdot\mathscr{A})\right]_i$$

$$= -\frac{\beta_D}{\nu}\hat{\mathscr{A}}_i + \frac{a^2}{10}\left[\nabla\times\left(n\nabla\times\hat{\mathscr{A}}\right) + n\nabla^2\hat{\mathscr{A}}\right]_i - \beta_D\nabla\psi, \tag{75}$$

which, together with the term $-\nabla\psi$ arising from the transformation of p_m, is readily seen to be consistent with Eq. (39).

The invariant part of the stress (70) can be decomposed into a traceless symmetric component S, an antisymmetric component A, and an isotropic component $-(p_m + q_m)\mathbf{I}$. In terms of the force multipoles defined before the symmetric part may be written as

$$\mathsf{S} = \beta_C(\langle\boldsymbol{\sigma}_C\rangle + \langle p_C\rangle\mathbf{I}) + \left(1 + \frac{a^2}{14}\nabla^2\right)(n\mathbf{t}^s) + \nabla\cdot(n\mathbf{s}^s) + \nabla\nabla{:}(n\mathbf{r}^s)$$

$$-\frac{a^2}{10}n\left[\nabla\mathscr{A} + (\nabla\mathscr{A})^{\mathrm{T}} - \frac{2}{3}\mathbf{I}(\nabla\cdot\mathscr{A})\right], \tag{76}$$

where the superscript T denotes the transpose. The antisymmetric part may be written as

$$\mathsf{A}_{ji} = \epsilon_{ijk}\mathsf{R}_k, \tag{77}$$

where the pseudo-vector R is defined by

$$\mathsf{R} = \frac{1}{2}\left[\overline{n\int_{|\mathbf{r}|=a}\mathrm{d}S_r(\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}} - \frac{1}{2}\nabla\cdot\left(\overline{n\int_{|\mathbf{r}|=a}\mathrm{d}S_r\mathbf{r}\left[(\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}\right]}\right)\right.$$

$$\left.+\frac{1}{6}\nabla\nabla{:}\left(\overline{n\int_{|\mathbf{r}|=a}\mathrm{d}S_r\mathbf{r}\mathbf{r}\left[(\boldsymbol{\sigma}_C\cdot\mathbf{n})\times\mathbf{r}\right]}\right)\right]. \tag{78}$$

Aside from the factor $\frac{1}{2}n$, the first term is just the average couple acting on the particles and is the dominant contribution to the antisymmetric part of the stress. With no inertia and without an external couple applied to the particles it would evidently vanish. But it is interesting to note that, even in these conditions, in general the stress contains a weak antisymmetric component of order (a^2/L^2) that would, however, vanish in the case of a spatially uniform system.

Finally, the isotropic part of the viscous stress is

$$q_{\rm m} = \frac{a^2}{5}\partial_k\big(n\mathscr{A}_k^*\big) - \frac{a^2}{14}\partial_k\partial_l\big(nt_{kl}^{\rm s}\big)^* + \frac{1}{15}a^2 n\nabla\cdot\mathscr{A} - \partial_k\big(ns_{kmm}^{\rm i}\big) - \partial_l\partial_k\big(nr_{lkmm}^{\rm i}\big), \tag{79}$$

where we write $\mathscr{A}^*$, $(t^{\rm s})^*$ to denote the parts of $\mathscr{A}$, $t^{\rm s}$ arising from the viscous stresses. We may thus write

$$\nabla\cdot\big(\beta_{\rm C}\langle\boldsymbol{\sigma}_{\rm C}\rangle + \beta_{\rm D}\mathscr{L}\big) - n\mathscr{A} = \nabla\cdot\big[-(p_{\rm m}+q_{\rm m})\mathbf{I} + \mathsf{S} + \mathsf{A}\big]$$

$$-\frac{\beta_{\rm D}}{v}\mathscr{A} + \frac{a^2}{10}\big[(\nabla n)\times(\nabla\times\mathscr{A}) + n\nabla(\nabla\cdot\mathscr{A})\big]. \tag{80}$$

Upon substituting Eq. (80) into Eq. (28), the complete viscous part of the stress defined in Eq. (33) is therefore found to be

$$\Sigma_{ji} = -q_{\rm m}\delta_{ji} + \mathsf{S}_{ji} + \mathsf{A}_{ji} + \frac{a^2}{10}\big(n\partial_i\mathscr{A}_j - \mathscr{A}_i\partial_j n\big) + (\Sigma_{\rm a})_{ji}, \tag{81}$$

which, by Eqs. (38) and (36), is readily seen to be invariant under a gauge transformation to order $(a/L)^4$.

6. Discussion

In the previous section we have been led to define the mixture pressure as in Eq. (73):

$$p_{\rm m} = \beta_{\rm C}\langle p_{\rm C}\rangle + \left(1+\frac{a^2}{10}\nabla^2\right)\big(nv\bar{p}^{\rm e}\big) + \frac{a^2}{5}\nabla\cdot\left(\overline{n\int_{|\mathbf{r}|=a}{\rm d}S_r(-p_{\rm C})\mathbf{n}}\right)$$

$$+\frac{a^2}{14}\nabla\nabla:\left[\overline{n\int_{|\mathbf{r}|=a}{\rm d}S_r\left(\mathbf{nn}-\frac{1}{3}\mathbf{I}\right)p_{\rm C}}\right]. \tag{82}$$

In the case of a mixture in equilibrium at rest $p^{\rm e} = p_{\rm C} = \langle p_{\rm C}\rangle$, all derivatives vanish and $p_{\rm m} = \langle p_{\rm C}\rangle$ as expected. For a flowing uniform mixture, while $\bar{p}^{\rm e} \neq \langle p_{\rm C}\rangle$ in general, all derivatives vanish so that

$$p_{\rm m} = \beta_{\rm C}\langle p_{\rm C}\rangle + nv\bar{p}^{\rm e}, \tag{83}$$

or, since in this case from Eq. (12) $\beta_{\rm D} = nv$,

$$p_{\rm m} = \beta_{\rm C}\langle p_{\rm C}\rangle + \beta_{\rm D}\bar{p}^{\rm e}. \tag{84}$$

Formal averaging procedures often lead to a mixture pressure defined by

$$p_{\rm m} = \beta_{\rm C}\langle p_{\rm C}\rangle + \beta_{\rm D}\langle p_{\rm D}\rangle, \tag{85}$$

where the physical meaning of $\langle p_{\rm D}\rangle$ may not be entirely clear unless the disperse phase is a fluid, as discussed in Section 1. The result (84) gives to this quantity a well-defined meaning in all cases. As a matter of fact, the analysis of Drew and Lahey (1993) based on elasticity theory for the uniform case shows that the isotropic part of $\langle \boldsymbol{\sigma}_{\rm D}\rangle$ is precisely equal to $\bar{p}^{\rm e}$.

If the finite extent of the particles is negligible, $a \simeq 0$, $v \simeq 0$, $\beta_{\rm C} \simeq 1$ and Eq. (82) gives

$$p_{\rm m} \simeq \langle p_{\rm C}\rangle, \tag{86}$$

which is also expected.

Suppose, again, that the differentiated terms in Eq. (82) are negligible. In this case, the continuous-phase momentum equation (28) may be written as

$$\mathbf{I}_{\rm C} = \beta_{\rm C}\nabla\cdot(-\langle p_{\rm C}\rangle\mathbf{I} + \boldsymbol{\Sigma}) - \left(\bar{p}^{\rm e} - \langle p_{\rm C}\rangle\right)\nabla\beta_{\rm D} - \mathbf{F}_{\rm C} - \beta_{\rm C}\nabla\psi_{\rm C}, \tag{87}$$

where

$$\mathbf{F}_{\rm C} = n\mathscr{A} - \beta_{\rm D}\nabla\cdot\left(-\bar{p}^{\rm e}\mathbf{I} + \boldsymbol{\Sigma}\right) + \nabla\cdot\boldsymbol{\Sigma}_{\rm a}, \tag{88}$$

may be considered the force exerted by the disperse on the continuous phase. The corresponding form of the disperse-phase equation would then be

$$\mathbf{I}_{\rm D} = \beta_{\rm D}\nabla\cdot(-\langle p_{\rm C}\rangle\mathbf{I} + \boldsymbol{\Sigma}) + \mathbf{F}_{\rm D} - \beta_{\rm C}\nabla\psi_{\rm C}, \tag{89}$$

with

$$\mathbf{F}_{\rm D} = n\mathscr{A} - \beta_{\rm D}\nabla\cdot(-\langle p_{\rm C}\rangle\mathbf{I} + \boldsymbol{\Sigma}) + \nabla\cdot\boldsymbol{\Sigma}_{\rm a}. \tag{90}$$

The equations are now written in terms of $\langle p_{\rm C}\rangle$ rather than $p_{\rm m}$, but this form requires a closure to express $\bar{p}^{\rm e}$ in terms of $\langle p_{\rm C}\rangle$.

It may also be noted that, again dropping the differentiated terms in Eq. (82),

$$\nabla p_{\rm m} = \beta_{\rm C}\nabla\langle p_{\rm C}\rangle + \left(\bar{p}^{\rm e} - \langle p_{\rm C}\rangle\right)\nabla\beta_{\rm D} + \beta_{\rm D}\nabla\bar{p}^{\rm e}. \tag{91}$$

The first two terms in the right-hand side appear in many averaged equations formulations (see, e.g., Ishii, 1975; Park et al., 1998). The last term was introduced in Prosperetti and Jones (1984) on the basis of a less rigorous argument than that leading to the more precise result (82).

All of the above considerations ignore the differentiated terms in Eq. (82). These may be viewed as corrections of progressively higher order in the ratio a/L of the particle radius to the macroscopic length L and may therefore be small in many cases. The form (84) would then be a good approximation to the mixture pressure.

Finally, since some steps of the derivation presented in the previous section are a matter of choice, it is appropriate to discuss to what extent the results are unique.

Uniqueness of the expression (82) for the pressure can be proven rather simply by following the same procedure as in the previous section but focusing exclusively on the pressure part of the stress (21) rather than carrying at the same time pressure and viscous components. Given Eqs. (45), (53) and (62), the calculation is straightforward and we omit the details.

The grouping of the remaining terms into symmetric, antisymmetric, and isotropic parts, however, is not unique. For example, on the basis of the identity

$$(\nabla n) \times (\nabla \times \mathscr{A}) + n\nabla(\nabla \cdot \mathscr{A}) = \nabla \times (n\nabla \times \mathscr{A}) + n\nabla^2 \mathscr{A}, \tag{92}$$

it would be possible to redefine R in the antisymmetric part of the stress as

$$\mathsf{R}' = \mathsf{R} + \frac{1}{10} a^2 n \nabla \times \mathscr{A}. \tag{93}$$

In this case, the terms in the last line of Eq. (80) become

$$-\frac{\beta_{\mathrm{D}}}{v}\mathscr{A} + \frac{a^2}{10} n \nabla^2 \mathscr{A}. \tag{94}$$

Formally, the new term in Eq. (93) could be interpreted as a source of antisymmetry at the macroscopic level associated with the mean hydrodynamic force on the particles. Clearly, this lack of uniqueness is a matter of interpretation and does not affect the physical content of the equation. Ultimately, a choice among the various possible forms must rest on matters of convenience and physical interpretation. It is worth noting that, in the case of spheres in Stokes flow all subject to the same force (e.g., gravity), $\mathscr{A}$ has to equal the applied force and is therefore a constant. In this case there would be no difference between the options described.

7. The trace of the mixture stress

Another route to the definition of a mixture pressure in a disperse two-phase flow might be to follow a standard procedure in Continuum Mechanics and use the definition

$$\tilde{p}_{\mathrm{m}} = -\frac{1}{3} \mathrm{Tr}\, \boldsymbol{\Sigma}_{\mathrm{T}}. \tag{95}$$

This prescription suffers, however, from several shortcomings. In the first place, since only $\nabla \cdot \boldsymbol{\Sigma}_{\mathrm{T}}$ has a physical significance, rather than $\boldsymbol{\Sigma}_{\mathrm{T}}$ itself, the physics would remain unaffected by the addition to $\boldsymbol{\Sigma}_{\mathrm{T}}$ of any arbitrary divergenceless, but not traceless, tensor, while the definition (95) would not. The tensor (69) used before is an example. Secondly, in the case of a single-phase compressible fluid, the definition (95) leads to the introduction of a 'mechanical' pressure different from the true thermodynamic pressure, the difference being associated with the volume viscosity of the fluid. In the case of the two-phase mixture considered here, while the mixture as a whole is incompressible, the individual phases are not in the sense that their volume fractions generally change in time and space. The quantity defined by Eq. (95) would therefore also contain irreversible contributions due to interphase slip that one would be reluctant to consider as parts of a properly defined pressure. An example is shown in Eq. (109). Finally, the definition (95) may be unsatisfactory on intuitive grounds: one would expect that the pressure in the mixture would in some way be related to the continuous-phase pressure, for which one has a strong physical intuition. The definition (21) shows however that $\boldsymbol{\Sigma}_T$ also contains a contribution from the internal stresses in the particles, the physical relevance of which for the average behavior of the suspension is less clear.

These are general comments. To better appreciate them it is interesting to look at the specific form taken by Eq. (95) when applied to the stress $\boldsymbol{\Sigma}_T$ as defined in Eq. (21):

$$-\frac{1}{3}\,\mathrm{Tr}\,\boldsymbol{\Sigma}_T = -\frac{1}{3}\beta_C\langle \mathrm{Tr}\,\boldsymbol{\sigma}_C\rangle - \frac{1}{3}\mathrm{Tr}\big(\beta_D\mathscr{L}[\boldsymbol{\sigma}_C]\big) - \frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_a, \tag{96}$$

where, from the expansion (7)

$$\mathrm{Tr}\big(\beta_D\mathscr{L}\big) = n\mathscr{T}_{jj} + \partial_k\big(n\mathscr{S}_{kjj}\big) + \partial_l\partial_k\big(n\mathscr{R}_{lkjj}\big) + \cdots. \tag{97}$$

The transformation property of this quantity under a gauge transformation is readily obtained by using the previous results (36), (46), (55) and (63):

$$-\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_T = -\frac{1}{3}\mathrm{Tr}\,\hat{\boldsymbol{\Sigma}}_T - \left[\beta_C + \left(1 + \frac{a^2}{18}\nabla^2\right)nv\right]\psi - \frac{14}{45}a^2 v(\nabla n)\cdot(\nabla\psi)$$
$$- \frac{a^4 v}{105}(\nabla\nabla n):(\nabla\nabla\psi). \tag{98}$$

It is evident that this is not the proper transformation of a physical quantity to be identified with a pressure.

8. Examples

To illustrate the results given in the previous sections it is useful to see their consequences in some specific examples. We consider the case of spheres in potential flow and at vanishing Reynolds number, for both of which general exact solutions of the flow at the particle level can be written down. These expressions require a knowledge of the 'incident' microscopic flow (p^i,$\mathbf{u}^i$) in the neighborhood of each particle, which is of course only possible by numerical means. General expressions in terms of these fields are given in Appendix C. As explained in Appendix D, closed-form expressions can be obtained in the dilute limit with $O(\beta_D)$ accuracy, and it is these expressions that we give here. In presenting these results it is convenient to assume $\psi = \psi_C$ so that body-force effects on continuous-phase quantities can be disregarded.

8.1. Potential flow

For potential flow problems, the boundary conditions to be applied to the averaged equations only involve the normal component of the velocity. Unless additional boundary conditions are formulated, it is therefore necessary to truncate the equations so as to avoid the appearance of spatial derivatives of velocities and pressure higher than the first. We thus give expressions limited to these terms. Higher order terms are shown in Appendix C.

The quantity p^e defined in Eq. (74) is found to be

$$p^{e} = \langle p_{C} \rangle - \frac{1}{4}\rho_{C}(\mathbf{w} - \langle \mathbf{u}_{m} \rangle) \cdot (\mathbf{w} - \mathbf{u}_{m}), \tag{99}$$

where

$$\mathbf{u}_{m} = \beta_{C}\mathbf{u}_{C} + \beta_{D}\langle \mathbf{u}_{D} \rangle, \tag{100}$$

is the mean volumetric flux (that, to the present order, can be equated to the continuous-phase average velocity $\langle \mathbf{u}_{C} \rangle$). Upon keeping only first-order spatial derivatives, with Eq. (99), the total mixture pressure p_{m} defined in Eq. (73) becomes

$$p_{m} = \langle p_{C} \rangle - \frac{1}{4}\rho_{C}\beta_{D}\left[(\bar{\mathbf{w}} - \mathbf{u}_{m}) \cdot (\bar{\mathbf{w}} - \mathbf{u}_{m}) - \mathrm{Tr}\,\mathbf{M}_{D}\right], \tag{101}$$

where

$$\mathbf{M}_{D} = \overline{\mathbf{w}\mathbf{w}} - \bar{\mathbf{w}}\bar{\mathbf{w}}, \tag{102}$$

is the Reynolds stress of the particle velocity field. The result (101), with $\mathbf{M}_{D} = 0$, has been suggested by several authors (Ishii, 1975; Drew, 1983; Biesheuvel and van Wijngaarden, 1984; Prosperetti and Jones, 1984, and others) not all of whom, however, realized its limited validity to first order in the volume fraction and first order in the ratio of the micro- to the macroscale a/L. The physical relevance of the quantity p_{m} as found here is highlighted by the result of Zhang and Prosperetti (1994b) according to which, on average, bubbles immersed in the flow considered here would respond to the fluctuations of p_{m} rather than those of $\langle p_{C} \rangle$.

To the same accuracy we find

$$\mathscr{T}_{0}[p_{C}\mathbf{I}] = -\frac{3}{20}\rho_{C}\left[(\bar{\mathbf{w}} - \mathbf{u}_{m}) \cdot (\bar{\mathbf{w}} - \mathbf{u}_{m}) - \mathrm{Tr}\,\mathbf{M}_{D}\right]\mathbf{I}$$

$$+\frac{9}{20}\rho_{C}\left[(\bar{\mathbf{w}} - \mathbf{u}_{m})(\bar{\mathbf{w}} - \mathbf{u}_{m}) - \mathbf{M}_{D}\right], \tag{103}$$

while the contributions of $\mathscr{S}$ and $\mathscr{R}$ give derivatives of higher order and can therefore be disregarded. The hydrodynamic particle force $\mathscr{A}$ contains contributions from 'virtual buoyancy', added mass, lift, and particle Reynolds stress (Zhang and Prosperetti, 1994a):

$$\mathscr{A} = -v\nabla p_{m} + \frac{1}{2}\rho_{C}\left[\frac{\partial \mathbf{u}_{m}}{\partial t} + \mathbf{u}_{m} \cdot \nabla \mathbf{u}_{m} - \frac{\partial \bar{\mathbf{w}}}{\partial t} - \bar{\mathbf{w}} \cdot \nabla \bar{\mathbf{w}} + \frac{1}{\beta_{D}}\nabla \cdot \left(\beta_{D}\mathbf{M}_{D}\right)\right]$$

$$+\frac{1}{2}\rho_{C}(\nabla \times \mathbf{u}_{m}) \times (\bar{\mathbf{w}} - \mathbf{u}_{m}). \tag{104}$$

Here $\mathscr{L}$ reduces to $\mathscr{T}$ and thus

$$-\frac{1}{3}\mathrm{Tr}\left(\beta_{D}\mathscr{L}[\boldsymbol{\sigma}_{C}]\right) = \langle p_{C} \rangle\left(1 + \frac{a^{2}}{18}\nabla^{2}\right)(nv). \tag{105}$$

Furthermore

$$-\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_{\mathrm{T}} = \langle p_{\mathrm{C}}\rangle\left[1-\frac{2}{45}a^2\nabla^2(nv)\right]-\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_{\mathrm{a}}, \tag{106}$$

which is remarkably different from Eq. (101).

8.2. Stokes flow

As a further example, consider the case of spherical rigid particles at negligibly small Reynolds number.[4] To first order in β_{D}, as shown in Appendix D, one may identify the average of the particle-level continuous-phase pressure with $\langle p_{\mathrm{C}}\rangle$ and therefore

$$p_{\mathrm{m}} = \beta_{\mathrm{C}}\langle p_{\mathrm{C}}\rangle + \left(1+\frac{a^2}{10}\nabla^2\right)(nv\langle p_{\mathrm{C}}\rangle), \tag{107}$$

or, if only first-order derivatives of the pressure are allowed in the final form of the momentum equation as before,

$$p_{\mathrm{m}} = \beta_{\mathrm{C}}\langle p_{\mathrm{C}}\rangle + \langle p_{\mathrm{C}}\rangle\left(1+\frac{a^2}{10}\nabla^2\right)(nv) = \langle p_{\mathrm{C}}\rangle. \tag{108}$$

In this case, therefore, the mixture pressure coincides with the continuous-phase pressure.

We can now allow for second-order derivatives of the velocity in the averaged equations. Thus, the trace of the stress becomes

$$-\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_{\mathrm{T}} = \langle p_{\mathrm{C}}\rangle\left[1-\frac{2}{45}a^2\nabla^2(nv)\right]+\frac{1}{4}\mu_{\mathrm{C}}\nabla\cdot\left[nv(\mathbf{u}_{\mathrm{m}}-\bar{\mathbf{w}})\right]-\frac{1}{3}\mathrm{Tr}\,\boldsymbol{\Sigma}_{\mathrm{a}}. \tag{109}$$

The second term is a contribution to the isotropic part of the stress wherever the flux of particles relative to the continuous phase has net sinks or sources. The presence of such velocity-dependent terms illustrates the previous statement about the difference between 'mechanical' pressure and mixture pressure.

Expressions for the other quantities arising in the analysis of Section 5 are also readily available for this case:

$$\mathscr{A} = 6\pi\mu_{\mathrm{C}}a\left(-\mathbf{u}_{\Delta}+\frac{a^2}{6}\nabla^2\mathbf{u}_{\mathrm{m}}\right), \tag{110}$$

$$\mathbf{t}^{\mathrm{s}} = 5v\mu_{\mathrm{C}}\mathbf{E}_{\mathrm{m}}, \tag{111}$$

$$\partial_k\left(ns^{\mathrm{s}}_{kji}\right) = \frac{3}{10}v\mu_{\mathrm{C}}(n\mathbf{E}_{\Delta}+\mathbf{E}_{\nabla}), \tag{112}$$

where we have defined

[4] Of course, the smallness of the particle Reynolds number does not imply the smallness of the mixture Reynolds number.

$$\mathbf{E}_\mathrm{m} = \frac{1}{2}\left[\nabla\mathbf{u}_\mathrm{m} + (\nabla\mathbf{u}_\mathrm{m})^\mathrm{T}\right], \tag{113}$$

$$\mathbf{E}_\Delta = \frac{1}{2}\left[\nabla\mathbf{u}_\Delta + (\nabla\mathbf{u}_\Delta)^\mathrm{T}\right] - \frac{1}{3}(\nabla\cdot\mathbf{u}_\Delta)\mathbf{I}, \tag{114}$$

$$\mathbf{E}_\nabla = \frac{1}{2}[\mathbf{u}_\Delta\nabla n + \nabla n\mathbf{u}_\Delta] - \frac{1}{3}(\mathbf{u}_\Delta\cdot\nabla n)\mathbf{I}, \tag{115}$$

in which

$$\mathbf{u}_\Delta = \bar{\mathbf{w}} - \mathbf{u}_\mathrm{m}, \tag{116}$$

is the 'slip' velocity. With these expressions, and recalling that, for a viscous incompressible fluid, one has the exact result (Zhang and Prosperetti, 1997)

$$\beta_\mathrm{C}\langle\boldsymbol{\sigma}_\mathrm{C}\rangle = -\beta_\mathrm{C}\langle p_\mathrm{C}\rangle + 2\mu_\mathrm{C}\mathbf{E}_\mathrm{m}, \tag{117}$$

we find

$$\mathsf{S} = \left(1 + \frac{5}{2}nv\right)\mu_\mathrm{C}\mathbf{E}_\mathrm{m} + \frac{6}{5}nv\mu_\mathrm{C}\mathbf{E}_\Delta + \frac{3}{10}v\mu_\mathrm{C}\mathbf{E}_\nabla. \tag{118}$$

The first term will be recognized as the correct Einstein effective viscosity of a dilute uniform suspension. The other terms vanish when the slip velocity vanishes, as often happens in a uniform suspension of density-matched spheres.

The vector R characterizing the antisymmetric part of the stress is

$$\mathsf{R} = 3v\mu_\mathrm{C}n\left(\frac{1}{2}\nabla\times\mathbf{u}_\mathrm{m} - \bar{\boldsymbol{\Omega}}\right) + \frac{3}{2}v\mu_\mathrm{C}\nabla\times\left[n(\mathbf{w} - \mathbf{u}_\mathrm{m})\right], \tag{119}$$

where $\bar{\boldsymbol{\Omega}}$ is the mean particle angular velocity, and will vanish in the absence of external couples. Finally, the isotropic part of the viscous stress is given by

$$q_\mathrm{m} = -\frac{3}{10}nv\mu_\mathrm{C}\nabla\cdot\mathbf{u}_\Delta + \frac{1}{5}v\mu_\mathrm{C}\nabla\cdot(n\mathbf{u}_\Delta), \tag{120}$$

and, as anticipated in Section 7, would be non-zero in the presence of a non-zero slip velocity.

9. A formulation of the momentum equations

The momentum equations in the form (28) and (29) are attractive in terms of their manifest symmetry and satisfaction of the action–reaction principle. Their usefulness is somewhat limited, however, by the fact that they explicitly involve the stress inside the particles.

While it is trivially true that the particles influence the motion of the continuous phase and, at a fundamental level, they do so through the stress field at their surface, the point we are discussing here is different. Consider for examples two suspensions of identical metal spheres.

Due to the fabrication process, the spheres in the first suspension have some residual stresses, while those in the second one do not. In the same microscopic flow field, the precise deformation of the pre-stressed spheres will be different from that of the other ones, and therefore the microscopic flow field will also be affected differently. Yet it would be difficult to argue that a practically useful description of the flow on the basis of averaged equations should be able to distinguish between the two situations. It would seem that, whenever it makes sense to approximate the particle behavior as rigid, useful averaged equations should not require further detailed information about the particle structure.

It is therefore interesting to examine the possibility of finding a formulation different from (28) and (29) — applicable to the case of particles sufficiently stiff to be modeled as rigid — that does not involve the particles' internal degrees of freedom. Such a formulation would also be useful for the purpose of closing averaged equations models as the necessity to account for the particle internal structure would further complicate an already very difficult task. If, for example, a closure is sought on the basis of direct numerical simulations, one would be forced to solve for the particle internal stresses in addition to the fluid motion with a very great and undesirable increase in computational effort.

For the continuous phase a formulation free of reference to the particle internal structure has already been presented in Eq. (13). However, there is much to be gained by reconsidering that equation in the light of the new definition of the mixture pressure given in Section 5 and of the covariance properties discussed at the end of that section. Let us define the continuous-phase viscous contribution to the mixture stress $\boldsymbol{\Sigma}_{\mathrm{C}}$ by

$$\boldsymbol{\Sigma}_{\mathrm{C}} = -q_{\mathrm{m}}\mathbf{I} + \mathsf{S} + \mathsf{A}. \tag{121}$$

By using Eq. (80) to express the terms in the right-hand side of Eq. (13) we therefore have

$$\mathbf{I}_{\mathrm{C}} = \nabla\cdot(-p_{\mathrm{m}}\mathbf{I} + \boldsymbol{\Sigma}_{\mathrm{C}}) - \frac{\beta_{\mathrm{D}}}{v}\mathscr{A} + \frac{a^2}{10}\left[(\nabla n)\times(\nabla\times\mathscr{A}) + n\nabla(\nabla\cdot\mathscr{A})\right] - \beta_{\mathrm{C}}\nabla\psi_{\mathrm{C}}. \tag{122}$$

By using the transformation properties of p_{m} and $\mathscr{A}$, i.e., $p_{\mathrm{m}} = \hat{p}_{\mathrm{m}} - \psi$, $\mathscr{A} = \hat{\mathscr{A}} + v\nabla\psi$, and recalling that $\boldsymbol{\Sigma}_{\mathrm{C}}$ is invariant, it is immediately seen that the quantity

$$\mathbf{f} = \frac{1}{v}\mathscr{A} - \nabla\cdot(-p_{\mathrm{m}}\mathbf{I} + \boldsymbol{\Sigma}_{\mathrm{C}}), \tag{123}$$

is also gauge invariant. This fact implies that a closure for $\mathbf{f}$ cannot include a term proportional to ∇p_{m} which, as shown in Marchioro et al. (1999b), is a very useful observation. It is interesting to rewrite this relation as

$$\mathscr{A} = v\nabla\cdot(-p_{\mathrm{m}}\mathbf{I} + \boldsymbol{\Sigma}_{\mathrm{C}}) + v\mathbf{f}, \tag{124}$$

which shows that the average hydrodynamic force per particle $\mathscr{A}$ is the result of two effects. The first one is the first term in the right-hand side, which would vanish for a uniform system; it evidently represents the force acting on the particles due to the structure of the flow over the macroscopic scale. This term is therefore responsible for the so-called 'virtual buoyancy' force and its generalization to the viscous component of the stress. The second term must therefore be due to the forces acting on the particle phase due to the local flow in their surroundings, such as added mass, lift, drag, and others. This decomposition of the force on the disperse phase, which is substantiated by the explicit dilute results of Section 8, was postulated earlier on intuitive grounds (Prosperetti and Jones, 1984), but is here rigorously deduced from the averaged equations.[5]

In terms of the continuous-phase 'Reynolds stress'

$$\mathbf{M}_C = \langle \mathbf{u}_C \rangle \langle \mathbf{u}_C \rangle - \langle \mathbf{u}_C \mathbf{u}_C \rangle, \tag{125}$$

the momentum equation (122) may be rewritten as

$$\rho_C \beta_C \left(\frac{\partial \langle \mathbf{u}_C \rangle}{\partial t} + \langle \mathbf{u}_C \rangle \cdot \nabla \langle \mathbf{u}_C \rangle \right) = -\beta_C \nabla \cdot (- p_m \mathbf{I} + \boldsymbol{\Sigma}_C) \beta_D \mathbf{f}$$

$$+ \rho_D \nabla \cdot \left(\beta_C \mathbf{M}_D \right) - \beta_C \nabla \psi_C + \frac{a^2}{10} \left[(\nabla n) \times (\nabla \times \mathscr{A}) + n \nabla (\nabla \cdot \mathscr{A}) \right]. \tag{126}$$

We now turn to the disperse-phase momentum equation (16). Following the same procedure outlined in the Appendix to Zhang and Prosperetti (1997), it can be readily shown that

$$\mathbf{I}_D - \nabla \cdot \boldsymbol{\Sigma}_a = \mathbf{I}_w + \left(\beta_D - nv \right) \nabla \psi_D, \tag{127}$$

where

$$\mathbf{I}_w = \rho_D \left[\frac{\partial}{\partial t} (nv \bar{\mathbf{w}}) + \nabla \cdot (nv \overline{\mathbf{w}\mathbf{w}}) \right]. \tag{128}$$

We can therefore rewrite the particle momentum equation (16) as

$$\mathbf{I}_w = n \mathscr{A} - nv \nabla \psi_D. \tag{129}$$

This form is evidently free of any reference to the particle internal stress. Alternatively, the equation may be written as

$$\rho_D nv \left(\frac{\partial \bar{\mathbf{w}}}{\partial t} + \bar{\mathbf{w}} \cdot \nabla \bar{\mathbf{w}} \right) = nv \nabla \cdot (- p_m \mathbf{I} + \boldsymbol{\Sigma}_C) + nv \mathbf{f} + \rho_D \nabla \cdot (nv \mathbf{M}_D) - nv \nabla \psi_D, \tag{130}$$

with $\mathbf{M}_D$ the particle Reynolds stress defined in Eq. (102). Eqs. (126) and (130) are remarkably similar in their structure to the completely symmetric forms (28) and (29). There are two differences however. The first one is that the disperse-phase equation is expressed in terms of nv rather than β_D. This circumstance arises because $\mathbf{I}_w$ is the rate of change of the momentum of all the particles the center of which is in the unit volume. In a non-uniform system, this quantity is evidently different from $\mathbf{I}_D$, the rate of change of the momentum of the particle material contained in the unit volume. The second difference is the last term in Eq. (126) which has no counterpart in the disperse-phase equation. This is due to the fact, proven in Section 3,

[5] Note that, for the Stokes flow example of Section 8, $\nabla \cdot (-p_m \mathbf{I} + \boldsymbol{\Sigma}_C) = O(\beta_D)$ and is, therefore, negligible in the expression for $\mathscr{A}$ given in Eq. (110).

that a perfectly symmetric form requires the introduction of the particle internal stress because, in a symmetric treatment of the phases, it is necessary to include the particle internal stress in the definition of the mixture stress. The slight loss of symmetry caused by the last term in Eq. (126) seems to be a worth-while price to pay to avoid the introduction of $\boldsymbol{\Sigma}_a$ in the averaged equations.

It is interesting to consider the special case in which particle inertia is negligible. The microscopic particle momentum equation (14) gives then $\nabla \cdot \boldsymbol{\sigma}_D = \nabla \psi_D$ from which, according to the definition (19) of $\boldsymbol{\Sigma}_a$, one can show that

$$\nabla \cdot \boldsymbol{\Sigma}_a = \frac{a^2}{10} v \nabla \psi_D (\nabla^2 n). \tag{131}$$

On the other hand, from Eq. (16), $\nabla \cdot \boldsymbol{\Sigma}_a = \beta_D \nabla \psi_D - n \mathscr{A}[\boldsymbol{\sigma}_C]$ and therefore, upon comparing,

$$\mathscr{A} = v \nabla \psi_D, \tag{132}$$

which simply states that the total hydrodynamic force on the particles is balanced by the body force. Upon substituting into the definition (27) of $\mathbf{F}$ we, therefore, find the interesting result

$$\mathbf{F} = \beta_D \mathbf{f}. \tag{133}$$

In this case, from Eq. (132), we also see that $\nabla \times \mathscr{A} = 0$, $\nabla \cdot \mathscr{A} = 0$, and therefore the two equations (126) and (130) acquire the desired form, with $\mathbf{f}$ identifiable as the true interphase force per unit particle volume. In this case the particle internal dynamics disappears completely from the formulation. Such a simple result would not hold when particle inertia is not negligible.

10. Summary and conclusions

The action of a conservative body force on an incompressible fluid can be equivalently considered as the effect of a modified pressure redefined to include the potential of the body force. By requiring that this fundamental property be satisfied by the mixture pressure in a disperse two-phase flow, we have been led to a unique, well-defined prescription for this quantity given in Eq. (73). While this result is similar to others proposed in the past, it differs in detail and also puts the earlier, often heuristic, derivations on a firmer ground. Furthermore we have also shown that, provided the particles can be modelled as rigid bodies, a consideration of the covariance properties of the averaged momentum equations for the disperse and continuous phases under the transformation of the pressure leads to a formulation that does not contain any reference to the internal stress of the particles, Eqs. (126) and (130). The equivalence of this formulation with the more common ones obtained by a direct averaging of the microscopic equations has been explicitly proven in Sections 3 and 9. At no point in the derivation was it necessary to assume that the continuous phase was Newtonian. Although only a disperse phase consisting of equal spheres was explicitly considered, a generalization to unequal spheres or particles of arbitrary shape does not present great difficulties.

The results of the analysis have been illustrated in the case of dilute suspensions subject to potential or Stokes flow in Section 8. When substituted into the general expressions for the stress and interphase force given in Eqs. (76)–(80), these results constitute an exact, closed-form expression for the averaged momentum equations in these conditions.

Another interesting consequence of the study is the explicit identification of the interphase force, and its natural decomposition into large-scale (including, e.g., virtual buoyancy) and small-scale (including, e.g., added mass) components. Since the small-scale component has been shown to be invariant under a transformation of the pressure, it follows that pressure gradients are not necessary for its closure.

In addition to a general understanding of the issues surrounding the proper formulation of the averaged momentum equations, the chief purpose of the paper is to put into a sharper focus the problem of closing the equations. The mixture pressure is a primary variable and does not need to be closed, while everything else does. Hence, it is important to separate the pressure from the rest of the stress before attempting a closure. Furthermore, if a closure is based on the results of a direct numerical simulation, the present formulation shows that it is unnecessary to solve for the internal particle stress, which is useful in limiting the necessary computational effort. An application of the present analysis to the closure problem of spatially non-uniform suspensions is given in Marchioro et al. (1999b).

The study has relied on the ideas and techniques of ensemble averaging that we have described in a series of earlier papers (Zhang and Prosperetti, 1994a, 1994b, 1997; Prosperetti, 1998). The chief elements of the method are summarized in the Appendices.

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Appendix A. Averaging relations

The phase averaging method used in this paper has been described in detail in a series of earlier studies (Zhang and Prosperetti, 1994a, 1994b, 1997; Prosperetti, 1998). It will be sufficient here to recall some definitions and results that are proven in the cited references. We start by considering the case of particles in inviscid potential flow. Conceptually, expressions for finite Reynolds number are immediate generalizations that will be described later.

The potential flow induced by N equal homogeneous spherical particles embedded in a prescribed deterministic flow is only a function of the particle centers' instantaneous positions, $\mathbf{y}^\alpha$, $\alpha = 1,2,\ldots,N$, and center-of-mass velocities $\mathbf{w}^\alpha$. The set of vectors $\{\mathbf{y}^\alpha,\mathbf{w}^\alpha\}$ constitutes a configuration $\mathscr{C}^N$ (or, more simply N) of the system under consideration. Many such systems, each one containing N particles in different configurations, constitute the statistical ensemble that we study. The probability density with which configurations occur in the ensemble is denoted by $P(N)$ and therefore

$$P(N)\mathrm{d}C^N \equiv P\left(\mathbf{y}^1,\mathbf{w}^1,\ldots,\mathbf{y}^N,\mathbf{w}^N\right)\mathrm{d}^3y^1\mathrm{d}^3w^1\cdots\mathrm{d}^3y^N\mathrm{d}^3w^N, \tag{A1}$$

represents the fraction of members of the ensemble such that the center of sphere 1 is in the elemental volume d^3y^1 around $\mathbf{y}^1$, etc. In view of the identity of the particles we use the normalization (see, e.g., Batchelor, 1972)

$$\int \mathrm{d}\mathscr{C}^N P(N) = N!. \tag{A2}$$

Let $\chi_{\mathrm{C,D}}(\mathbf{x};N)$ be the characteristic function for the continuous (subscript C) or disperse (subscript D) phase. For the present case of equal spheres of radius a an explicit representation of χ_{D} is

$$\chi_{\mathrm{D}}(\mathbf{x};N) = \sum_{\alpha=1}^{N} H\left(a - |\mathbf{x}-\mathbf{y}^\alpha|\right), \tag{A3}$$

where H is Heaviside's distribution, and $\chi_{\mathrm{C}} = 1-\chi_{\mathrm{D}}$. The phase volume fractions $\beta_{\mathrm{C,D}}$ are given by

$$\beta_{\mathrm{C,D}}(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\chi_{\mathrm{C,D}}(\mathbf{x};N), \tag{A4}$$

while the particle number density n is given by

$$n(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\sum_{\alpha=1}^{N}\delta\left(\mathbf{x}-\mathbf{y}^\alpha\right)$$

$$= \frac{1}{(N-1)!}\int \mathrm{d}^3w^1\int \mathrm{d}^3y^{(2)}\int \mathrm{d}^3y^{(3)}\cdots\int \mathrm{d}^3y^{(N)}P\left(\mathbf{x},\mathbf{y}^{(2)},\ldots,\mathbf{y}^{(N)}\right), \tag{A5}$$

where the second step follows from the identity of the particles.

If the averaged quantities vary slowly over distances comparable to the particle radius, by carrying out a Taylor series expansion in the integrand of (A4), one readily proves the relation (12) between β_{D} and nv given in the text.

The phase average of any field $f_{\mathrm{C,D}}$ pertaining to either phase is given by

$$\beta_{\mathrm{C,D}}(\mathbf{x})\langle f_{\mathrm{C,D}}\rangle(\mathbf{x}) = \frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\chi_{\mathrm{C,D}}(\mathbf{x};N)f_{\mathrm{C,D}}(\mathbf{x};N), \tag{A6}$$

where $f_{\mathrm{C,D}}(\mathbf{x};N)$ denotes the exact (microscopic) value of f in the appropriate phase at a point $\mathbf{x}$ when the spheres are in the configuration N. Note that, in general, each configuration evolves in time, although we do not indicate time dependence explicitly.

In the following, we also will need conditional probability densities. The one-particle probability distribution $P(\mathbf{y},\mathbf{w})$ is defined by

$$P(1) \equiv P(\mathbf{y},\mathbf{w}) = \frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}^{N-1}P(N), \tag{A7}$$

where the integration is over the degrees of freedom of particles $2,3,\ldots,N$. With this quantity, we can define the conditional probability $P(N-1|1)$ by $P(N)=P(1)P(N-1|1)$ and the one-particle conditional averages by

$$\beta_C^1\langle f_C\rangle_1(\mathbf{x},t|\mathbf{y},\mathbf{w})=\frac{1}{(N-1)!}\int \mathrm{d}\mathscr{C}^{N-1}\chi_C f_C P(N-1|1), \tag{A8}$$

where β_C^1, the conditional volume fraction, is given by this same relation with $f_C=1$.

For the disperse phase, in addition to the phase average $\langle f_D\rangle$, we shall make use of a particle average defined as follows. Let g^α be a quantity pertaining to the α-th particle as a whole, such as the velocity of the center of mass, the angular velocity, etc. Then we define

$$n(\mathbf{x})\bar{g}(\mathbf{x})=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\left[\sum_{\alpha=1}^{N}\delta\left(\mathbf{x}-\mathbf{y}^\alpha\right)g^\alpha(N)\right] \tag{A9}$$

$$=\frac{1}{(N-1)!}\int \mathrm{d}^3w\int \mathrm{d}\mathscr{C}^{N-1}P(\mathbf{x},\mathbf{w},N-1)g^1(\mathbf{x},\mathbf{w},N-1), \tag{A10}$$

where the notation $g^\alpha(N)$ indicates that the value of g for particle α is, in general, dependent on the position of all the particles and the expression in the second line is a consequence of the identity of the particles. Keeping in mind the explicit expression (A3) of χ_D, we note an analogy between the definitions of β_D, n, and $\langle f_D\rangle,\bar{g}$.

Upon assuming the fluid to be incompressible, averaging of the microscopic equation of continuity leads to

$$\frac{\partial\beta_C}{\partial t}+\nabla\cdot\left(\beta_C\langle\mathbf{u}_C\rangle\right)=0, \tag{A11}$$

and, similarly,

$$\frac{\partial\beta_D}{\partial t}+\nabla\cdot\left(\beta_D\langle\mathbf{u}_D\rangle\right)=0. \tag{A12}$$

Since the matrix–sphere interface has zero measure, $\beta_C+\beta_D=1$ and we thus find adding (A11) and (A12)

$$\nabla\cdot\mathbf{u}_m=0, \tag{A13}$$

where the mean volume flux $\mathbf{u}_m$ is defined in Eq. (100) or, from Eq. (A6),

$$\mathbf{u}_m=\frac{1}{N!}\int \mathrm{d}\mathscr{C}^N P(N)\left[\chi_C(\mathbf{x};N)\mathbf{u}_C(\mathbf{x};N)+\chi_D(\mathbf{x};N)\mathbf{u}_D(\mathbf{x};N)\right]. \tag{A14}$$

Expressing the conservation of particle number requires the particle average $\bar{\mathbf{w}}$ of the center-of-mass velocity $\mathbf{w}$ defined according to Eq. (A9):

$$\frac{\partial n}{\partial t}+\nabla\cdot(n\bar{\mathbf{w}})=0. \tag{A15}$$

In addition to potential flow, another situation in which the exact microscopic flow is only a function of the particle variables is Stokes flow, in which case, however, the probability density does not depend on the particle velocities. The equations appropriate for this situation can be obtained from those given before simply by omitting the integrations over particle velocities. As a consequence, it is immediate to deduce from Eqs. (A5) and (A7) that $P(1) = n$.

For the general case of finite Reynolds numbers (as well as for inviscid, but not necessarily irrotational flows) one must consider, in addition to the particle degrees of freedom, the degrees of freedom of the continuous phase. If $\mathbf{Q}$ denotes collectively these degrees of freedom, we would now have $P(N) = P(\mathbf{y}^{\alpha},\mathbf{w}^{\alpha},\mathbf{Q})$, but all the relations given before would be applicable provided all the integrals are understood to include an integration over $\mathbf{Q}$.

For brevity, in the paper we have only indicated explicitly the integration over the spatial variables, leaving the integrations over velocities and continuous-phase degrees of freedom implied. Thus, the equations in the form given are literally correct only in the case of Stokes flow. For example, if the integration over the velocity of particle 1 were indicated, Eq. (5) would become

$$\beta_{\mathrm{C}}\langle\nabla\cdot\boldsymbol{\sigma}_{\mathrm{C}}\rangle = \nabla\cdot\big(\beta_{\mathrm{C}}\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle\big) - \int_{|\mathbf{x}-\mathbf{y}|=a} \mathrm{d}S_y \int \mathrm{d}^3 w P(\mathbf{y},\mathbf{w})\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x}|\mathbf{y},\mathbf{w})\cdot\mathbf{n}_y. \tag{A16}$$

Appendix B. The small-particle approximation

We derive here the expansions (6) and (19) that constitute the small-particle approximation.

For the first result, the starting point is the relation (5) between the average of the divergence and the divergence of the average. To approximate the integral term in this relation we observe that, in the bulk of the suspension, $\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x}|\mathbf{y})$ varies much more rapidly with respect to the variable $\mathbf{x}$ than with respect to the particle center $\mathbf{y}$. Let then $\mathbf{r} = \mathbf{x} - \mathbf{y}$ be a vector with length a directed from the particle center $\mathbf{y}$ to the point $\mathbf{x}$ on the particle surface and define, omitting non-essential variables,

$$F(\mathbf{r},\mathbf{y}) = P(\mathbf{y})\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x}|\mathbf{y}). \tag{B1}$$

In view of the slowness of the dependence on $\mathbf{y}$, we expand F in a Taylor series centered at $\mathbf{x}$:

$$F(\mathbf{r},\mathbf{y}) = F(\mathbf{r},\mathbf{x}) - \mathbf{r}\cdot\nabla_y\big\{F(\mathbf{r},\mathbf{y}) - \mathbf{r}\cdot\nabla_y\big[F(\mathbf{r},\mathbf{y}) - \mathbf{r}\cdot\nabla_y\big(F(\mathbf{r},\mathbf{y}) + \cdots\big)\big]\big\}, \tag{B2}$$

with all the derivatives taken with respect to the variable $\mathbf{y}$ and then evaluated at $\mathbf{y} = \mathbf{x}$. Upon substituting this relation into (5), we have a result of the form (6) with

$$\mathscr{A}[\boldsymbol{\sigma}_{\mathrm{C}}] = \int_{|\mathbf{r}|=a} \mathrm{d}S\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x}+\mathbf{r}|\mathbf{x})\cdot\mathbf{n}, \tag{B3}$$

$$\mathscr{T}[\boldsymbol{\sigma}_{\mathrm{C}}] = a\int_{|\mathbf{r}|=a} \mathrm{d}S\mathbf{n}\big[\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1(\mathbf{x}+\mathbf{r}|\mathbf{x})\cdot\mathbf{n}\big], \tag{B4}$$

$$\mathscr{S}[\boldsymbol{\sigma}_{\rm C}] = -\frac{1}{2}a^2 \int_{|\mathbf{r}|=a} {\rm d}S\mathbf{nn}\big[\langle\boldsymbol{\sigma}_{\rm C}\rangle_1(\mathbf{x}+\mathbf{r}|\mathbf{x})\cdot\mathbf{n}\big], \tag{B5}$$

$$\mathscr{R}[\boldsymbol{\sigma}_{\rm C}] = \frac{1}{6}a^3 \int_{|\mathbf{r}|=a} {\rm d}S\mathbf{nnn}\big[\langle\boldsymbol{\sigma}_{\rm C}\rangle_1(\mathbf{x}+\mathbf{r}|\mathbf{x})\cdot\mathbf{n}\big], \tag{B6}$$

and so on. It can readily be shown, by using the definitions (A8) and (A9), that these expressions can equivalently be written as in Eqs. (8)–(11).

To prove Eq. (19) we proceed similarly. From the definition of conditional and unconditional phase averages, using the identity of the particles, it is easy to show that

$$\beta_{\rm D}\langle\nabla\cdot\boldsymbol{\sigma}_{\rm D}\rangle = \int_{|\mathbf{x}-\mathbf{y}|\le a} {\rm d}^3 y n(\mathbf{y})\langle\nabla\cdot\boldsymbol{\sigma}_{\rm D}\rangle_1(\mathbf{x}|\mathbf{y}), \tag{B7}$$

where the integral is extended to all the particle centers such that $\mathbf{x}$ is inside the particle phase. As before, we observe that the integrand varies slowly with respect to the variable $\mathbf{y}$ so that we can carry out a Taylor series expansion similar to (B2) to find

$$\beta_{\rm D}\langle\nabla\cdot\boldsymbol{\sigma}_{\rm D}\rangle = n(\mathbf{x})\int_{|\mathbf{r}|\le a} {\rm d}^3 r\langle\nabla_{\rm r}\cdot\boldsymbol{\sigma}_{\rm D}\rangle_1(\mathbf{x}+\mathbf{r}|\mathbf{x}) + \nabla\cdot\boldsymbol{\Sigma}_{\rm a}, \tag{B8}$$

where $\nabla_{\rm r}$ denotes the gradient with respect to the distance from the particle center and $\boldsymbol{\Sigma}_{\rm a}$ is defined in Eq. (19). The first term in the right-hand side of Eq. (B8) can be further manipulated by interchanging the conditional averaging and the integration over the particle volume and then applying the divergence theorem to write it as an integral over the particle surface. Here the exact, microscopic condition of normal stress continuity applies, $\boldsymbol{\sigma}_{\rm C}\cdot\mathbf{n} = \boldsymbol{\sigma}_{\rm D}\cdot\mathbf{n}$, and therefore the integral in Eq. (B8) is the same as the one as (B3) defining $\mathscr{A}[\boldsymbol{\sigma}_{\rm C}]$. With this remark, (15) follows.

Appendix C. Relations for potential flow and Stokes flow

We derive here the results given in Section 8 taking advantage of the fact that, for potential and Stokes flow, general solutions are available for the flow fields. Here we first show the general expressions, that must be evaluated numerically. Their dilute-limit form is discussed further in Appendix D and shown in Section 8.

We write ($p^{\rm i}$,$\mathbf{u}^{\rm i}$) to denote the 'incident' microscopic field in the neighborhood of the particle (defined, more precisely, as the part of the flow fields regular at the particle centers), and (p,$\mathbf{u}$) to denote the complete flow fields near the particles.

C.1. Potential flow

Consider a spherical particle instantaneously centered at $\mathbf{y}$ and translating with velocity $\mathbf{w}$ in a general (microscopic) flow field. Let the microscopic 'incident' flow ($\mathbf{u}^{\rm i}$,$p^{\rm i}$) in the neighborhood of the particle have a potential Φ so that

$$\mathbf{u}^{\rm i} = \nabla\Phi^{\rm i}, \tag{C1}$$

and write $\Phi^{\rm i}$ as

$$\Phi^{\rm i} = \sum_{l=0}^{\infty} \frac{1}{l!} (\mathbf{x}-\mathbf{y})^{(l)} \nabla^{(l)} \Phi^{\rm i}(\mathbf{y}), \tag{C2}$$

where the notation is such that

$$(\mathbf{x}-\mathbf{y})^{(l)} \nabla^{(l)} \Phi^{\rm i}(\mathbf{y}) = \left[(x-y)_i (x-y)_j \cdots (x-y)_k \partial_i \partial_j \cdots \partial_k\right] \Phi^{\rm i}, \tag{C3}$$

with all derivatives evaluated at the particle center. The particle modifies the incident flow to a disturbed flow ($\mathbf{u}$,p) with potential

$$\mathbf{u} = \nabla \Phi, \tag{C4}$$

where, in order to satisfy the kinematic boundary condition at the particle surface (Zhang and Prosperetti, 1994a),

$$\Phi = -\frac{1}{2} \frac{a^3}{|\mathbf{x}-\mathbf{y}|^3} \mathbf{w} \cdot (\mathbf{x}-\mathbf{y}) + \sum_{l=0}^{\infty} \frac{1}{l!} \left[1 + \frac{l}{l+1} \left(\frac{a}{|\mathbf{x}-\mathbf{y}|}\right)^{2l+1}\right] (\mathbf{x}-\mathbf{y})^{(l)} \nabla^{(l)} \Phi^{\rm i}(\mathbf{y}). \tag{C5}$$

To calculate the pressure field we need to take time derivatives of the potentials at a fixed position $\mathbf{x}$. For this purpose we need to remember that Φ and $\Phi^{\rm i}$ depend on time both explicitly, through $\mathbf{w}$ and the spatial derivatives of $\Phi^{\rm i}$, and implicitly through $\mathbf{y}$, so that

$$\left(\frac{\partial \Phi}{\partial t}\right)_x = \left(\frac{\partial \Phi}{\partial t}\right)_y - \mathbf{w} \cdot \nabla \Phi, \tag{C6}$$

with a similar expression for $(\partial \Phi^{\rm i}/\partial t)_x$. From Eqs. (C1) and (C14) one finds, at the particle surface,

$$\frac{p^{\rm i}}{\rho_{\rm C}} = C(t) - \frac{\partial \Phi^{\rm i}}{\partial t} - \frac{1}{2} \mathbf{u}^{\rm i} \cdot \mathbf{u}^{\rm i}, \tag{C7}$$

$$\begin{aligned} \frac{p}{\rho_{\rm C}} = {} & \frac{p^{\rm i}}{\rho_{\rm C}} + \frac{9}{8}\left[\mathbf{n} \cdot (\mathbf{w}-\mathbf{u}^{\rm i})\right]^2 - \frac{5}{8}(\mathbf{w}-\mathbf{u}^{\rm i}) \cdot (\mathbf{w}-\mathbf{u}^{\rm i}) + \frac{1}{2} a \mathbf{n} \cdot \left(\dot{\mathbf{w}} - \frac{\partial \mathbf{u}^{\rm i}}{\partial t}\right) - \frac{1}{3} a^2 \mathbf{nn}{:}\frac{\partial \mathbf{e}^{\rm i}}{\partial t} + \\ & a\left[(\mathbf{I}-\mathbf{nn}) \cdot (\mathbf{w}-\mathbf{u}^{\rm i})\right] \mathbf{n}{:}\mathbf{e}^{\rm i} + \frac{9}{32} a^2 \left[(\mathbf{I}-\mathbf{nn}) \cdot (\mathbf{w}-\mathbf{u}^{\rm i})\right] \cdot (\mathbf{nn}{:}\nabla\nabla\nabla\Phi) \\ & -\frac{1}{18} a^2 \left[5(\mathbf{n} \cdot \mathbf{e}^{\rm i} \cdot \mathbf{n})^2 + 4(\mathbf{n} \cdot \mathbf{e}^{\rm i}) \cdot (\mathbf{n} \cdot \mathbf{e}^{\rm i})\right] + \cdots, \end{aligned} \tag{C8}$$

where

$$\mathbf{e}^{\rm i}(\mathbf{x}|N) = \frac{1}{2}\left[\nabla \mathbf{u}^{\rm i} + (\nabla \mathbf{u}^{\rm i})^{\rm T}\right], \tag{C9}$$

is the rate-of-strain tensor of the incident field evaluated at the particle center $\mathbf{x}$ when the particles have the configuration $\mathscr{C}^N$ and the dots stand for terms involving higher order spatial derivatives of $\mathbf{u}^{\mathrm{i}}$. Here as elsewhere we omit the explicit indication of non-essential variables such as time.

With this expression we find

$$p^{\mathrm{e}}(\mathbf{x}|N) = p^{\mathrm{i}}(\mathbf{x}|N) - \frac{1}{4}\rho_{\mathrm{C}}\left[\mathbf{w} - \mathbf{u}^{\mathrm{i}}(\mathbf{x}|N)\right]\cdot\left[\mathbf{w} - \mathbf{u}^{\mathrm{i}}(\mathbf{x}|N)\right] - \frac{5}{18}a^2\rho_{\mathrm{C}}\mathbf{e}^{\mathrm{i}}(\mathbf{x}|N){:}\mathbf{e}^{\mathrm{i}}(\mathbf{x}|N) + \cdots, \tag{C10}$$

The integral appearing in the definition (50) of $\mathscr{T}_0[p_{\mathrm{C}}\mathbf{I}]$ is

$$\frac{1}{\rho_{\mathrm{C}}}\int_{|\mathbf{r}|=a} \mathrm{d}S_r\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right)p_{\mathrm{C}} = -\frac{3}{20}v\left(\mathbf{w} - \mathbf{u}_0^{\mathrm{i}}\right)\cdot\left(\mathbf{w} - \mathbf{u}_0^{\mathrm{i}}\right)\mathbf{I} + \frac{9}{20}v\left(\mathbf{w} - \mathbf{u}_0^{\mathrm{i}}\right)\left(\mathbf{w} - \mathbf{u}_0^{\mathrm{i}}\right)$$

$$-a^2 v\left(\frac{1}{3}\frac{\partial}{\partial t} - \frac{3}{10}\mathbf{u}_0^{\mathrm{i}}\cdot\nabla + \frac{19}{30}\mathbf{w}\cdot\nabla\right)\mathbf{e}_0^{\mathrm{i}} - \frac{5}{21}va^2\mathbf{e}_0^{\mathrm{i}}\cdot\mathbf{e}_0^{\mathrm{i}} + \frac{5}{63}va^2\left(\mathbf{e}_0^{\mathrm{i}}{:}\mathbf{e}_0^{\mathrm{i}}\right)\mathbf{I} + \cdots, \tag{C11}$$

where, as before, the omitted terms involve higher order derivatives of $\mathbf{u}^{\mathrm{i}}$ and the subscript zero indicates values at the particle center. It is clear that this quantity is independent of the continuous-phase pressure as expected on the basis of the discussion in Section 5. The integral in the definition (58) of $\mathscr{S}_0[p_{\mathrm{C}}\mathbf{I}]$ is

$$\frac{1}{\rho_{\mathrm{C}}}\int_{|\mathbf{r}|=a} \mathrm{d}S\left(n_i n_j n_k - \frac{1}{5}\delta_{ijkl}n_l\right)p_{\mathrm{C}} = \frac{1}{5}v\delta_{ijkl}\left(\mathbf{u}_0^{\mathrm{i}} - \mathbf{w}\right)\cdot\nabla(u_{0l}^{\mathrm{i}} + \frac{1}{14}v\delta_{ijklmn}\left(u_{0l}^{\mathrm{i}} - w_l\right)e_{0mn}^{\mathrm{i}}$$

$$+\frac{1}{10}\delta_{jk}v\left(\dot{\mathbf{w}} - \frac{\partial\mathbf{u}_0^{\mathrm{i}}}{\partial t} - \mathbf{u}_0^{\mathrm{i}}\cdot\nabla\mathbf{u}_0^{\mathrm{i}}\right)_i - \frac{1}{5}v\delta_{jk}\left(\partial_i p^{\mathrm{i}}\right)_0, \tag{C12}$$

where δ_{ijkl} and δ_{ijklmn} are the completely symmetric isotropic tensors of ranks 4 and 6. Again, this quantity is seen to be independent of p_{C}. For the specification of p_{m} we also need the following result:

$$\frac{1}{\rho_{\mathrm{C}}}\int_{|\mathbf{r}|=a} \mathrm{d}S_{\mathrm{r}}\mathbf{n}p_{\mathrm{C}} = \frac{1}{2}v\left(\dot{\mathbf{w}} - \frac{\partial\mathbf{u}_0^{\mathrm{i}}}{\partial t} - \mathbf{u}_0^{\mathrm{i}}\cdot\nabla\mathbf{u}_0^{\mathrm{i}}\right) - v\left(\nabla p^{\mathrm{i}}\right)_0. \tag{C13}$$

C.2. Stokes flow

According to Lamb's general solution of the Stokes equations (Lamb, 1932), the incident flow fields in the neighborhood of a particle centered at the origin may be written as

$$p^{\mathrm{i}}(\mathbf{x}) = \mu_{\mathrm{C}}\sum_{n=0}^{\infty}p_n(\mathbf{x}), \tag{C14}$$

$$\mathbf{u}^{\mathrm{i}}(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+3)} \left[\frac{1}{2}(n+3) r^2 \nabla p_n - n \mathbf{x} p_n \right] + \sum_{n=0}^{\infty} \left[\nabla \phi_n + \nabla \times (\mathbf{x} \chi_n) \right], \tag{C15}$$

where $r = |\mathbf{x}|$ and the p_n, ϕ_n, and χ_n are solid harmonics of order n. The introduction into this flow of a particle centered at the origin and having a translational velocity $\mathbf{w}$ and an angular velocity $\mathbf{\Omega}$ causes a disturbance given by

$$p^{\mathrm{d}} = \mu_{\mathrm{C}} \sum_{n=1}^{\infty} p_{-n-1}, \tag{C16}$$

$$\mathbf{u}^{\mathrm{d}} = \sum_{n=1}^{\infty} \frac{1}{n(2n-1)} \left[-\frac{1}{2}(n-2) r^2 \nabla p_{-n-1} + (n+1) \mathbf{x} p_{-n-1} \right]$$

$$+ \sum_{n=1}^{\infty} \left[\nabla \phi_{-n-1} + \nabla \times (\mathbf{x} \chi_{-n-1}) \right], \tag{C17}$$

where

$$p_{-n-1} = -\frac{1}{2} \frac{2n-1}{n+1} n \left[p_n + \frac{2}{a^2}(2n+1)\phi_n \right] \left(\frac{a}{r} \right)^{2n+1} + \frac{3}{2} \mathbf{w} \cdot \mathbf{x} \frac{1}{r^2} \delta_{n1}, \tag{C18}$$

$$\phi_{-n-1} = -\frac{a^2}{4} \frac{n}{n+1} \left[\frac{2n+1}{2n+3} p_n + \frac{2}{a^2}(2n-1)\phi_n \right] \left(\frac{a}{r} \right)^{2n+1} + \frac{a^2}{4} \mathbf{w} \cdot \mathbf{x} \frac{1}{r^2} \delta_{n1}, \tag{C19}$$

$$\chi_{-n-1} = -\left(\frac{a}{r} \right)^{2n+1} \chi_n + \mathbf{\Omega} \cdot \mathbf{x} \frac{a^2}{r^2} \delta_{n1}. \tag{C20}$$

With these relations it is easy to prove that, at the particle surface,

$$p^{\mathrm{d}} + p^{\mathrm{i}} = \frac{3}{2a} \mu_{\mathrm{C}} \mathbf{w} \cdot \mathbf{n} - \mu_{\mathrm{C}} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \left[\frac{1}{2}(n-2) p_n + \frac{n}{a^2}(2n-1)\phi_n \right], \tag{C21}$$

$$\begin{aligned} \boldsymbol{\sigma}_{\mathrm{C}} \cdot \mathbf{n} &= -\left(p^{\mathrm{d}} + p^{\mathrm{i}}\right)\mathbf{n} + \mu_{\mathrm{C}} \left[\nabla\left(\mathbf{u}^{\mathrm{d}} + \mathbf{u}^{\mathrm{i}}\right) + \left(\nabla\left(\mathbf{u}^{\mathrm{d}} + \mathbf{u}^{\mathrm{i}}\right)\right)^{\mathrm{T}} \right] \cdot \mathbf{n} \\ &= -\frac{3\mu_{\mathrm{C}}}{2a} \mathbf{w} - 3\mu_{\mathrm{C}} \mathbf{\Omega} \times \mathbf{n} + \mu_{\mathrm{C}} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \left[\frac{1}{2} a \nabla p_n - \mathbf{n} p_n \right] + \frac{\mu_{\mathrm{C}}}{a} \sum_{n-1}^{\infty} \frac{4n^2-1}{n+1} \nabla \phi_n \\ &\quad + \mu_{\mathrm{C}} \sum_{n=1}^{\infty} (2n+1) \nabla \chi_n \times \mathbf{n}, \end{aligned} \tag{C22}$$

where the p_n, ϕ_n, and χ_n are evaluated at the particle surface $r = a$.

The required surface integrals can be evaluated from these relations with results expressed in terms of the functions p_n, ϕ_n, χ_n evaluated at the particle center. However, these fields can readily be related to the incident field. For example, one finds

$$\left(\nabla\phi_1\right)_0 = \mathbf{u}_0^{\mathrm{i}}, \qquad \left(\partial_j\partial_k\phi_2\right)_0 = \frac{1}{2}\left(\partial_j u_k^{\mathrm{i}} + \partial_k u_j^{\mathrm{i}}\right)_0, \tag{C23}$$

and so on.

Using these results, one finds

$$p^{\mathrm{e}}(\mathbf{x}|N) = p^{\mathrm{i}}(\mathbf{x}|N), \tag{C24}$$

where the incident pressure $p^{\mathrm{i}}(\mathbf{x}|N)$ is evaluated at the particle center $\mathbf{x}$, and

$$a\int_{|\mathbf{r}|=a} \mathrm{d}S_r\left(\mathbf{nn} - \frac{1}{3}\mathbf{I}\right)p_{\mathrm{C}}(\mathbf{x}+\mathbf{r}|N) = -v\left[\nabla\mathbf{u}_0^{\mathrm{i}} + \left(\nabla\mathbf{u}_0^{\mathrm{i}}\right)^{\mathrm{T}}\right], \tag{C25}$$

which, again, is independent of the pressure as expected. Similarly,

$$S_{kji} = \frac{3}{4}v\mu\delta_{jk}\left[w_i - u_{0i}^{\mathrm{e}}\right] + \frac{1}{8}va^2\left[\delta_{ij}\left(\partial_k p^{\mathrm{e}}\right)_0 + \delta_{ik}\left(\partial_j p^{\mathrm{e}}\right)_0\right]$$

$$-\frac{1}{8}va^2\mu\left[5\partial_k\partial_j u_i^{\mathrm{e}} + \partial_i\left(\partial_j u_k^{\mathrm{e}} + \partial_k u_j^{\mathrm{e}}\right)\right]_0 - \frac{1}{16}va^4\left(\partial_i\partial_j\partial_k p^{\mathrm{e}}\right)_0. \tag{C26}$$

Appendix D. The dilute limit

The results of Appendix C lead directly to expressions for the average fields that have an $O(\beta_{\mathrm{D}})$ accuracy. The argument is the following.

As shown by Eqs. (B3)–(B6), in order to calculate $\mathscr{A}[\boldsymbol{\sigma}_{\mathrm{C}}]$ and $\beta_{\mathrm{D}}\mathscr{L}[\boldsymbol{\sigma}_{\mathrm{C}}]$ to first order in the disperse-phase volume fraction, it is evidently sufficient to know $\langle\boldsymbol{\sigma}_{\mathrm{C}}\rangle_1$ correct to zero order in β_{D}, i.e., as if no other particle were present. In other words, we are led to solve the problem

$$\nabla\cdot\langle\mathbf{u}_{\mathrm{C}}\rangle_1 = 0, \tag{D1}$$

$$\rho_{\mathrm{C}}\left(\frac{\partial}{\partial t}\langle\mathbf{u}_{\mathrm{C}}\rangle_1 + \langle\mathbf{u}_{\mathrm{C}}\rangle_1\cdot\nabla\langle\mathbf{u}_{\mathrm{C}}\rangle_1\right) = -\nabla\langle p_{\mathrm{C}}\rangle_1 + \mu_{\mathrm{C}}\nabla^2\langle\mathbf{u}_{\mathrm{C}}\rangle_1 + \rho_{\mathrm{C}}\mathbf{g}. \tag{D2}$$

The microscopic boundary condition on the normal velocity component at a point $\mathbf{x}$ on the surface of particle 1 is $\mathbf{u}_{\mathrm{C}}(\mathbf{y}^1 + \mathbf{r}|N)\cdot\mathbf{n}^1 = \mathbf{w}^1\cdot\mathbf{n}^1$, where $\mathbf{x} = \mathbf{y}^1 + \mathbf{r}$ with $\mathbf{r} = a\mathbf{n}^1$. In order to transform this condition into one concerning $\langle\mathbf{u}_{\mathrm{C}}\rangle_1$, we take the conditional average according to the definition (A8) noting that, since the condition is to be applied on the exterior surface of the particle, $\beta_{\mathrm{D}}^1 = 1$, $\chi_{\mathrm{C}} = 1$:

$$\mathbf{n}^1 \cdot \langle \mathbf{u}_C \rangle_1 \left(\mathbf{x} | \mathbf{y}^1, \mathbf{w}^1 \right) = \frac{1}{(N-1)!} \int d\mathscr{C}^{N-1} \mathbf{n}^1 \cdot \mathbf{u}_C P\left(N-1 | \mathbf{y}^1, \mathbf{w}^1 \right)$$

$$= \frac{1}{(N-1)!} \int d\mathscr{C}^{N-1} \mathbf{n}^1 \cdot \mathbf{w}^1 P\left(N-1 | \mathbf{y}^1, \mathbf{w}^1 \right). \tag{D3}$$

In the case of Stokes flow velocities are not integration variables and we can set $P(N-1|1) = P(N)/n$ to find, according to the definition of particle average (A10),

$$\mathbf{n} \cdot \langle \mathbf{u}_C \rangle_1 = \mathbf{n} \cdot \bar{\mathbf{w}}. \tag{D4}$$

For potential flow, or at finite Reynolds numbers, $\mathbf{w}^1$ is an independent variable in the phase space and is not affected by the integration. The remaining expression can then be evaluated to 1 given the normalization of $P(N-1|1)$. Thus, in this case (omitting the superscript 1), the boundary condition is

$$\mathbf{n} \cdot \langle \mathbf{u}_C \rangle_1 (\mathbf{x} | \mathbf{y}, \mathbf{w}) = \mathbf{n} \cdot \mathbf{w}. \tag{D5}$$

As the distance from the center of the particle, measured on the scale of the particle radius, gets large, the effect of the particle decreases and therefore the proper boundary condition is

$$\langle \mathbf{u}_C \rangle_1 (\mathbf{x} | \mathbf{y}, \mathbf{w}) \rightarrow \langle \mathbf{u}_C \rangle (\mathbf{x}) \quad \text{as} \quad |\mathbf{x} - \mathbf{y}| \rightarrow \infty \tag{D6}$$

As posed in (D1), (D2), (D6) and (D4) or (D5), at this level of accuracy the problem to be solved is that of a single sphere in a prescribed flow. The solutions for potential and Stokes flow given in Appendix C can readily be adapted to this case simply by observing that, in view of the boundary condition (D6), the incident fields are to be interpreted as the unconditionally averaged fields $(\langle \mathbf{u}_C \rangle, \langle p_C \rangle)$. Since $\mathbf{u}_m - \langle \mathbf{u}_C \rangle = O(\beta_D)$, at the same level of accuracy, we can also write $\mathbf{u}_m$ in place of $\langle \mathbf{u}_C \rangle$.

Note also that the previous argument implies that the particle averages of the incident fields evaluated at the particle centers can, to $O(\beta_D)$, be identified with the unconditional averages of the continuous-phase fields. This fact is sometimes called the Foldy approximation (see, e.g., Caflisch et al., 1985) with reference to a study by Foldy (1945) in which it first appeared; See also Hinch (1977).

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